矩形薄板弹性弯曲问题的一般解析解法

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提　要　本文对求解矩形薄板弹性弯曲问题采用先建立微分方程的一般解，然后根据问题的边界条件确定积分常数，这样求解比采用迭加法求解要简单容易。

一、基本方程的解

如图1所示矩形薄板弹性弯曲的基本方程为[1]

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2 \partial y^2}+\frac{\partial^4 W}{\partial y^4}=\frac{q}{D} \tag{1}$$

当四边为简支时可取双正弦级数解[2]

$$W=\sum_{m=1}\sum_{n=1}A_{mn}\sin\alpha x\sin\beta y \tag{2}$$

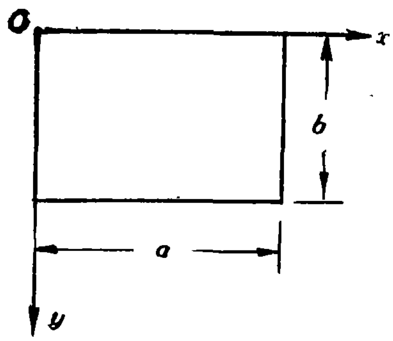
式中

$$A_{mn}=\frac{4\int_0^a\int_0^b q\sin\alpha x\sin\beta y dxdy}{Dab(\alpha^2+\beta^2)^2} \tag{3}$$

$$\alpha=\frac{m\pi}{a}\qquad m=1,2,\cdots\cdots \tag{4}$$

$$\beta=\frac{n\pi}{b}\qquad n=1,2,\cdots\cdots \tag{5}$$

等式(1)也可取单三角级数解以及一些代数多项式的解，或其他形式的解，对于各种不同边界条件的问题一般采用结构力学的迭加法来求解[3]，即将板看成是在实际荷载作用下的四边简支板；板边在力矩作用下的简支矩形板；三边简支另一边具有挠度（即自由边在剪力作用下）时的矩形板以及角点在具有挠度（反力作用下）时的矩形板，这样迭加而成，求解过程十分复杂。



本文建议先建立基本微分方程(1)的一般解，然后根据边界条件直接进行求解，这样避免了繁琐的迭加问题，方程(1)的一般解可写成不同的形式，本文取

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$$W=\sum_{m=1}(A_m \operatorname{ch}\alpha y+B_m \alpha y \operatorname{sh}\alpha y+C'_m \operatorname{sh}\alpha y+D'_m \alpha y \operatorname{ch}\alpha y)\sin\alpha x$$
$$+\sum_{n=1}(E_n \operatorname{ch}\beta x+F_n \beta x \operatorname{sh}\beta x+G'_n \operatorname{sh}\beta x+H'_n \beta x \operatorname{ch}\beta x)\sin\beta y$$
$$+a_{00}+a_{10}\frac{x}{a}+a_{01}\frac{y}{b}+a_{11}\frac{xy}{ab}+a_{20}\frac{x^2}{a^2}+a_{02}\frac{y^2}{b^2}+a_{21}\frac{x^2 y}{a^2 b}$$
$$+a_{12}\frac{ay^2}{ab^2}+a_{30}\frac{x^3}{a^3}+a_{03}\frac{y^3}{b^3}+a_{31}\frac{x^3 y}{a^3 b}+a_{13}\frac{xy^3}{ab^3}+W_0 \tag{6}$$

W_0为等式(1)的任一特解，本文取等式(2)，直接应用上式来求解将出现困难，文献[3]已指出，但本文采取：

$$\left.\begin{aligned}C'_m&=C_m/\operatorname{sh}\alpha b-A_m \operatorname{cth}\alpha b\\D'_m&=D_m/\operatorname{sh}\alpha b-B_m \operatorname{cth}\alpha b\\G'_n&=G_n/\operatorname{sh}\beta a-E_n \operatorname{cth}\beta a\\H'_n&=H_n/\operatorname{sh}\beta a-F_n \operatorname{cth}\beta a\end{aligned}\right\} \tag{7}$$

则避免了求解时的困难。

二、边界条件

板的各种边界条件分别为

$$(W)_{x=0},\ (W)_{x=a},\ (W)_{y=0},\ (W)_{y=b}$$
$$\left(\frac{\partial W}{\partial x}\right)_{x=0},\ \left(\frac{\partial W}{\partial x}\right)_{x=a},\ \left(\frac{\partial W}{\partial y}\right)_{y=0},\ \left(\frac{\partial W}{\partial y}\right)_{y=b}$$
$$(M_x)_{x=0}=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)_{x=0}$$
$$(M_x)_{x=a}=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)_{x=a}$$
$$(M_y)_{y=0}=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)_{y=0}$$
$$(M_y)_{y=b}=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)_{y=b}$$
$$(V_x)_{x=0}=-D\left[\frac{\partial^3 W}{\partial x^3}+(2-\nu)\frac{\partial^3 W}{\partial x\partial y^2}\right]_{x=0}$$
$$(V_x)_{x=a}=-D\left[\frac{\partial^3 W}{\partial x^3}+(2-\nu)\frac{\partial^3 W}{\partial x\partial y^2}\right]_{x=a}$$
$$(V_y)_{y=0}=-D\left[\frac{\partial^3 W}{\partial y^3}+(2-\nu)\frac{\partial^3 W}{\partial x^2\partial y}\right]_{y=0}$$
$$(V_y)_{y=b}=-D\left[\frac{\partial^3 W}{\partial y^3}+(2-\nu)\frac{\partial^3 W}{\partial x^2\partial y}\right]_{y=b}$$

将等式(6)代入以上各式并应用到(2)、(7)式和

$$1=\sum_{m=1}\frac{2(1-\cos m\pi)}{m\pi}\sin\alpha x$$

$$\frac{x}{a}=\sum_{m=1}\frac{2\cos m\pi}{m\pi}\sin\alpha x$$

$$\frac{x^2}{a^2}=-\sum_{m=1}\left[\frac{2\cos m\pi}{m\pi}+\frac{4(1-\cos m\pi)}{(m\pi)^3}\right]\sin\alpha x$$

$$\frac{x^3}{a^3}=-\sum_{m=1}\left[\frac{2\cos m\pi}{m\pi}-\frac{12\cos m\pi}{(m\pi)^3}\right]\sin\alpha x$$

$$\mathrm{ch}\beta x=-\sum_{m=1}\frac{2\alpha}{a(\alpha^2+\beta^2)}(\mathrm{ch}\beta a\cos m\pi-1)\sin\alpha x$$

$$\mathrm{sh}\beta x=-\sum_{m=1}\frac{2\alpha}{a(\alpha^2+\beta^2)}\mathrm{sh}\beta a\cos m\pi\sin\alpha x$$

$$\beta x\,\mathrm{ch}\beta x=-\sum_{m=1}\frac{2\alpha}{a(\alpha^2+\beta^2)}\Bigl(\beta a\,\mathrm{ch}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\mathrm{sh}\beta a\Bigr)\cos m\pi\sin\alpha x$$

$$\beta x\,\mathrm{sh}\beta x=-\sum_{m=1}\frac{2\alpha}{a(\alpha^2+\beta^2)}\Bigl[\beta a\,\mathrm{sh}\beta a\cos m\pi-\frac{2\beta^2}{\alpha^2+\beta^2}(\mathrm{ch}\beta a\cos m\pi-1)\Bigr]\sin\alpha x$$

以及将以上各式中的 x，a，m，α，β 分别改为 y，b，n，β，α 而得的等式，最后可以求得：

$$(W)_{x=0}=\sum_{n=1}\Bigl\{E_n+a_{00}\frac{2(1-\cos n\pi)}{n\pi}-a_{01}\frac{2\cos n\pi}{n\pi}-a_{02}\left[\frac{2\cos n\pi}{n\pi}+\frac{4(1-\cos n\pi)}{(n\pi)^3}\right]-a_{03}\left[\frac{2\cos n\pi}{n\pi}-\frac{12\cos n\pi}{(n\pi)^3}\right]\Bigr\}\sin\beta y \tag{8}$$

$$\begin{aligned}(W)_{x=a}=-\sum_{n=1}\Bigl\{&F_n\frac{\beta a}{\mathrm{sh}\beta a}-G_n-H_n\beta a\,\mathrm{cth}\beta a\\&-(a_{00}+a_{10}+a_{20}+a_{30})\frac{2(1-\cos n\pi)}{n\pi}+(a_{01}+a_{11}+a_{21}+a_{31})\frac{2\cos n\pi}{n\pi}\\&+(a_{02}+a_{12})\left[\frac{2\cos n\pi}{n\pi}+\frac{4(1-\cos n\pi)}{(n\pi)^3}\right]\\&+(a_{03}+a_{13})\left[\frac{2\cos n\pi}{n\pi}-\frac{12\cos n\pi}{(n\pi)^3}\right]\Bigr\}\sin\beta y\end{aligned} \tag{9}$$

$$(W)_{y=0}=\sum_{m=1}\Big\{A_m+a_{00}\frac{2(1-\cos m\pi)}{m\pi}-a_{10}\ \frac{2\cos m\pi}{m\pi}$$

$$-a_{20}\left[\frac{2\cos m\pi}{m\pi}+\frac{4(1-\cos m\pi)}{(m\pi)^3}\right]$$

$$-a_{30}\left[\frac{2\cos m\pi}{m\pi}-\frac{12\cos m\pi}{(m\pi)^3}\right]\Big\}\sin\alpha x \qquad (10)$$

$$(W)_{y=b}=-\sum_{m=1}\Big\{\beta_m\frac{\alpha b}{\mathrm{sh}\alpha b}-C_m-D_m\alpha b\,\mathrm{cth}\,\alpha b$$

$$-(a_{00}+a_{01}+a_{02}+a_{03})\ \frac{2(1-\cos m\pi)}{m\pi}+(a_{10}+a_{11}+a_{12}+a_{13})\ \frac{2\cos m\pi}{m\pi}$$

$$+(a_{20}+a_{21})\left[\frac{2\cos m\pi}{m\pi}+\frac{4(1-\cos m\pi)}{(m\pi)^3}\right]$$

$$+(a_{30}+a_{31})\left[\frac{2\cos m\pi}{m\pi}-\frac{12\cos m\pi}{(m\pi)^3}\right]\Big\}\sin\alpha x \qquad (11)$$

$$\left(\frac{\partial W}{\partial x}\right)_{x=0}=\sum_{n=1}\Big\{\sum_{m=1}\frac{2\alpha\beta}{b(\alpha^2+\beta^2)}\Big\{A_m+B_m\Big(\frac{\alpha b\cos n\pi}{\mathrm{sh}\alpha b}-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)$$

$$-C_m\cos n\pi-D_m\Big(\alpha b\,\mathrm{cth}\,\alpha b-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\cos n\pi\Big\}$$

$$-\beta\Big(E_n\mathrm{cth}\beta a+F_n\mathrm{cth}\beta a-\frac{G_n}{\mathrm{sh}\beta a}-\frac{H_n}{\mathrm{sh}\beta a}\Big)$$

$$+\frac{2}{n\pi a}\Big\{a_{10}(1-\cos n\pi)-a_{11}\cos n\pi-a_{12}\Big[\cos n\pi+\frac{2(1-\cos n\pi)}{(n\pi)^2}\Big]$$

$$-a_{13}\Big[1-\frac{6}{(n\pi)^2}\Big]\cos n\pi\Big\}+\sum_{m=1}A_{mn}a\Big\}\sin\beta y \qquad (12)$$

$$\left(\frac{\partial W}{\partial x}\right)_{x=a}=\sum_{n=1}\Big\{\sum_{m=1}\frac{2\alpha\beta\cos m\pi}{b(\alpha^2+\beta^2)}\Big\{A_m+B_m\Big(\frac{\alpha b\cos n\pi}{\mathrm{sh}\alpha b}-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)$$

$$-C_m\cos n\pi-D_m\Big(\alpha b\,\mathrm{cth}\,\alpha b-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\cos n\pi\Big\}$$

$$-\beta\Big[\frac{E_n}{\mathrm{sh}\beta a}+\frac{F_n}{\mathrm{sh}\beta a}-G_n\mathrm{cth}\beta a-H_n(\mathrm{cth}\beta a+\beta a)\Big]$$

$$+\frac{2}{n\pi a}\Big\{(a_{10}+2a_{20}+3a_{30})(1-\cos n\pi)-(a_{11}+2a_{21}+3a_{31})\cos n\pi$$

$$-a_{12}\Big[\cos n\pi+\frac{2(1-\cos n\pi)}{(n\pi)^2}\Big]-a_{13}\Big[1-\frac{6}{(n\pi)^2}\Big]\cos n\pi\Big\}$$

$$+\sum_{m=1}A_{mn}a\cos m\pi\Big\}\sin\beta y \qquad (13)$$

$$\left(\frac{\partial W}{\partial y}\right)_{y=0}=-\sum_{m=1}\left\{\alpha\left(A_m\operatorname{cth}\alpha b+B_m\operatorname{cth}\alpha b-\frac{C_m}{\operatorname{sh}\alpha b}-\frac{D_m}{\operatorname{sh}\alpha b}\right)\right.$$

$$-\sum_{n=1}\frac{2\alpha\beta}{a(\alpha^2+\beta^2)}\left\{E_n+F_n\left(\frac{\beta a\cos m\pi}{\operatorname{sh}\beta a}-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\right.$$

$$\left.-G_n\cos m\pi-H_m\left(\beta a\operatorname{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\cos m\pi\right\}$$

$$-\frac{2}{m\pi b}\left\{a_{01}(1-\cos m\pi)-a_{11}\cos m\pi-a_{21}\left(\cos m\pi+\frac{2(1-\cos m\pi)}{(m\pi)^2}\right)\right.$$

$$\left.\left.-a_{31}\left[1-\frac{6}{(m\pi)^2}\right]\cos m\pi\right\}-\sum_{n=1}A_{mn}\beta\right\}\sin\alpha x \tag{14}$$

$$\left(\frac{\partial W}{\partial y}\right)_{y=b}=-\sum_{m=1}\left\{\alpha\left[\frac{A_m}{\operatorname{sh}\alpha b}+\frac{B_m}{\operatorname{sh}\alpha b}-C_m\operatorname{cth}\alpha b-D_m(\operatorname{cth}\alpha b+\alpha b)\right]\right.$$

$$-\sum_{n=1}\frac{2\alpha\beta\cos n\pi}{a(\alpha^2+\beta^2)}\left\{E_n+F_n\left(\frac{\beta a\cos m\pi}{\operatorname{sh}\beta a}-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\right.$$

$$\left.-G_n\cos m\pi-H_n\left(\beta a\operatorname{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\cos m\pi\right\}$$

$$-\frac{2}{m\pi b}\left\{(a_{0.}+2a_{02}+3a_{03})(1-\cos m\pi)-(a_{11}+2a_{12}+3a_{13})\cos m\pi\right.$$

$$-a_{21}\left[\cos m\pi+\frac{2(1-\cos m\pi)}{(m\pi)^2}\right]$$

$$\left.\left.-a_{31}\left[1-\frac{6}{(m\pi)^2}\right]\cos m\pi\right\}-\sum_{n=1}A_{mn}\beta\cos n\pi\right\}\sin\alpha x \tag{15}$$

$$(M_x)_{x=0}=-D\sum_{n=1}\left\{\beta^2[E_n(1-\nu)+F_n2]\right.$$

$$+\frac{4}{n\pi}\left[a_{20}\frac{1-\cos n\pi}{a^2}+a_{02}\frac{\nu(1-\cos n\pi)}{b^2}\right.$$

$$\left.\left.-a_{21}\frac{\cos n\pi}{a^2}-a_{03}\frac{3\nu\cos n\pi}{b^2}\right]\right\}\sin\beta y \tag{16}$$

$$(M_x)_{x=a}=D\sum_{n=1}\left\{\beta^2\left\{F_n(1-\nu)\frac{\beta a}{\operatorname{sh}\beta a}-G_n(1-\nu)-\right.\right.$$

$$\left.-H_n[2+(1-\nu)\beta a\operatorname{cth}\beta a]\right\}-\frac{4}{n\pi}\left[(a_{20}+3a_{30})\frac{1-\cos n\pi}{a^2}\right.$$

$$+(a_{02}+a_{12})\frac{\nu(1-\cos n\pi)}{b^2}-(a_{21}+3a_{31})\frac{\cos n\pi}{a^2}$$

$$\left.\left.-(a_{03}+a_{13})\frac{3\nu\cos n\pi}{b^2}\right]\right\}\sin\beta y \tag{17}$$

$$(M_y)_{y=0}=-D\sum_{m=1}\Big\{\alpha^2(A_m(1-\nu)+B_m2]$$

$$+\frac{4}{m\pi}\Big[a_{02}\frac{1-\cos m\pi}{b^2}+a_{20}\frac{\nu(1-\cos m\pi)}{a^2}-a_{12}\frac{\cos m\pi}{b^2}$$

$$-a_{30}\frac{3\nu\cos m\pi}{a^2}\Big\}\sin\alpha x \tag{18}$$

$$(M_y)_{y=b}=D\sum_{m=1}\Big\{\alpha^2\Big\{\beta_m(1-\nu)\frac{\alpha b}{\operatorname{sh}\alpha b}-C_m(1-\nu)-D_m[2+(1-\nu)\alpha b\operatorname{cth}\alpha b]\Big\}$$

$$-\frac{4}{m\pi}\Big[(a_{02}+3a_{03})\frac{1-\cos m\pi}{b^2}+(a_{20}+a_{21})\frac{\nu(1-\cos m\pi)}{a^2}$$

$$-(a_{12}+3a_{13})\frac{\cos m\pi}{b^2}-(a_{30}+a_{31})\frac{3\nu\cos m\pi}{a^2}\Big]\Big\}\sin\alpha x \tag{19}$$

$$(V_x)_{x=0}=-D\sum_{n=1}\Big\{\sum_{m=1}\frac{2\alpha^3\beta}{b(\alpha^2+\beta^2)}\Big\{A_m(1-\nu)$$

$$+B_m\Big[2(2-\nu)+(1-\nu)\Big(\frac{\alpha b\cos n\pi}{\operatorname{sh}\alpha b}-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\Big]-C_m(1-\nu)\cos n\pi$$

$$-D_m\Big[2(2-\nu)+(1-\nu)\Big(\alpha b\operatorname{cth}\alpha b-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\Big]\cos n\pi\Big\}$$

$$+\beta^3\Big[E_n(1-\nu)\operatorname{cth}\beta a-F_n(1+\nu)\operatorname{cth}\beta a-G_n\frac{1-\nu}{\operatorname{sh}\beta a}$$

$$+H_n\frac{1+\nu}{\operatorname{sh}\beta a}\Big]+\frac{4}{n\pi}\Big[a_{30}\frac{3(1-\cos n\pi)}{a^3}+a_{12}\frac{(2-\nu)(1-\cos n\pi)}{ab^2}$$

$$-a_{31}\frac{3\cos n\pi}{a^3}-a_{13}\frac{3(2-\nu)\cos n\pi}{b^3}\Big]-\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2]\Big\}\sin\beta y \tag{20}$$

$$(V_x)_{x=a}=-D\sum_{n=1}\Big\{\sum_{m=1}\frac{2\alpha^3\beta\cos m\pi}{b(\alpha^2+\beta^2)}\Big\{A_m(1-\nu)$$

$$+B_m\Big[2(2-\nu)+(1-\nu)\Big(\frac{\alpha b\cos n\pi}{\operatorname{sh}\alpha b}-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\Big]$$

$$-C_m(1-\nu)\cos n\pi-D_m\Big[2(2-\nu)+(1-\nu)\Big(\alpha b\operatorname{cth}\alpha b-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\Big]\cos n\pi\Big\}$$

$$+\beta^3\Big\{E_n\frac{1-\nu}{\operatorname{sh}\beta a}-F_n\frac{1+\nu}{\operatorname{sh}\beta a}-G_n(1-\nu)\operatorname{cth}\beta a+H_n[(1+\nu)\operatorname{cth}\beta a$$

$$-(1-\nu)\beta a]\Big\}+\frac{4}{n\pi}\Big[a_{30}\frac{3(1-\cos n\pi)}{a^3}+a_{12}\frac{(2-\nu)(1-\cos n\pi)}{ab^2}$$

$$-a_{31}\frac{3\cos n\pi}{a^3}-a_{13}\frac{3(2-\nu)\cos n\pi}{ab^2}\Big]-\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2]\cos m\pi\Big\}\sin\beta y \tag{21}$$

$$(V_y)_{y=0}=-D\sum_{m=1}\Big\{\alpha^3\Big[A_m(1-\nu)\mathrm{cth}\alpha b-B_m(1+\nu)\mathrm{cth}\alpha b$$

$$-C_m\frac{1-\nu}{\mathrm{sh}\alpha b}+D_m\frac{1+\nu}{\mathrm{sh}\alpha b}\Big]+\sum_{n=1}\frac{2\alpha\beta^3}{a(\alpha^2+\beta^2)}\Big\{E_n(1-\nu)$$

$$+F_n\Big[2(2-\nu)+(1-\nu)\Big(\frac{\beta a\cos m\pi}{\mathrm{sh}\beta a}-\frac{2\beta^2}{\alpha^2+\beta^2}\Big)\Big]-G_n(1-\nu)\cos m\pi$$

$$-H_n\Big[2(2-\nu)+(1-\nu)\Big(\beta a\mathrm{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\Big)\Big]\cos m\pi\Big\}$$

$$+\frac{4}{m\pi}\Big[a_{03}\frac{3(1-\cos m\pi)}{b^3}+a_{21}\frac{(2-\nu)(1-\cos m\pi)}{a^2b}-a_{13}\frac{3\cos m\pi}{b^3}$$

$$-a_{31}\frac{3(2-\nu)\cos m\pi}{a^3}\Big]-\sum_{n=1}A_{mn}\beta[\beta^2+(2-\nu)\alpha^2]\Big\}\sin\alpha x \tag{22}$$

$$(V_y)_{y=b}=-D\sum_{m=1}\Big\{\alpha^3\Big\{A_m\frac{1-\nu}{\mathrm{sh}\alpha b}-B_m\frac{1+\nu}{\mathrm{sh}\alpha b}$$

$$-C_m(1-\nu)\mathrm{cth}\alpha b+D_m[(1+\nu)\mathrm{cth}\,\alpha b-(1-\nu)\alpha b]\Big\}$$

$$+\sum_{n=1}\frac{2\alpha\beta^3\cos n\pi}{a(\alpha^2+\beta^2)}\Big\{E_n(1-\nu)+F_n\Big[2(2-\nu)+(1-\nu)\Big(\frac{\beta a\cos m\pi}{\mathrm{sh}\beta a}$$

$$-\frac{2\beta^2}{\alpha^2+\beta^2}\Big)\Big]-G_n(1-\nu)\cos m\pi-H_n\Big[2(2-\nu)+(1-\nu)$$

$$\cdot\Big(\beta a\mathrm{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\Big)\Big]\cos m\pi\Big\}+\frac{4}{m\pi}\Big[a_{03}\frac{3(1-\cos m\pi)}{b^3}$$

$$+a_{21}\frac{(2-\nu)(1-\cos m\pi)}{a^2b}-a_{13}\frac{3\cos m\pi}{b^3}-a_{31}\frac{3(2-\nu)\cos m\pi}{a^2b}\Big]$$

$$-\sum_{n=1}A_{mn}\beta[\beta^2+(2-\nu)\alpha^2)\cos n\pi\Big\}\sin\alpha x \tag{23}$$

板的两个角点条件分别由

$$W,\ \frac{\partial W}{\partial x},\ \frac{\partial W}{\partial y}$$

$$M_x=-D\Big(\frac{\partial^2W}{\partial x^2}+\nu\frac{\partial^2W}{\partial y^2}\Big)$$

$$M_y=-D\Big(\frac{\partial^2W}{\partial y^2}+\nu\frac{\partial^2W}{\partial x^2}\Big)$$

$$R=2D(1-\nu)\frac{\partial^2W}{\partial x\partial y}.$$

来确定，将等式(2)和(7)代入(6)，然后代入以上各式并分别令

$$x=0,\ y=0$$
$$x=a,\ y=0$$
$$x=0,\ y=b$$
$$x=a,\ y=b$$

可以求得

$$W_{(0,0)}=a_{00} \tag{24}$$

$$W_{(a,0)}=a_{00}+a_{10}+a_{20}+a_{30} \tag{25}$$

$$W_{(0,b)}=a_{00}+a_{01}+a_{02}+a_{03} \tag{26}$$

$$W_{(a,b)}=a_{00}+a_{10}+a_{01}+a_{11}+a_{20}+a_{02}+a_{21}+a_{12}+a_{30}+a_{03}+a_{31}+a_{13} \tag{27}$$

$$\frac{\partial W}{\partial x}_{(0,0)}=\sum_{m=1}\alpha A_m+a_{10}\frac{1}{a} \tag{28}$$

$$\frac{\partial W}{\partial x}_{(a,0)}=\sum_{m=1}\alpha A_m\cos m\pi+(a_{10}+2a_{20}+3a_{30})\frac{1}{a} \tag{29}$$

$$\frac{\partial W}{\partial x}_{(0,b)}=-\sum_{m=1}\alpha\left(B_m\frac{\alpha b}{\mathrm{sh}\alpha b}-C_m-D_m\alpha b\,\mathrm{cth}\alpha b\right)+(a_{10}+a_{11}+a_{12}+a_{13})\frac{1}{a} \tag{30}$$

$$\frac{\partial W}{\partial x}_{(a,b)}=-\sum_{m=1}\alpha\left(B_m\frac{\alpha b}{\mathrm{sh}\alpha b}-C_m-D_m\alpha b\,\mathrm{cth}\,\alpha b\right)\cos m\pi+(a_{10}+a_{11}+2a_{20}+2a_{21}+a_{12}+3a_{30}+3a_{31}+a_{13})\frac{1}{a} \tag{31}$$

$$\frac{\partial W}{\partial y}_{(0,0)}=\sum_{n=1}\beta E_n+a_{01}\frac{1}{b} \tag{32}$$

$$\frac{\partial W}{\partial y}_{(a,0)}=-\sum_{n=1}\beta\left(F_n\frac{\beta a}{\mathrm{sh}\beta a}-G_n-H_n\beta a\,\mathrm{cth}\beta a\right)+(a_{01}+a_{11}+a_{21}+a_{31})\frac{1}{b} \tag{33}$$

$$\frac{\partial W}{\partial y}_{(0,b)}=\sum_{n=1}\beta E_n\cos n\pi+(a_{01}+2a_{02}+3a_{03})\frac{1}{b} \tag{34}$$

$$\frac{\partial W}{\partial y}_{(a,b)}=-\sum_{n=1}\beta\left(F_n\frac{\beta a}{\mathrm{sh}\beta a}-G_n-H_n\beta a\,\mathrm{cth}\beta a\right)\cos n\pi+(a_{01}+a_{11}+2a_{02}+2a_{12}+a_{21}+3a_{03}+a_{31}+3a_{13})\frac{1}{b} \tag{35}$$

$$(M_x)_{(0,0)}=-D\left(a_{20}\frac{2}{a^2}+a_{02}\frac{2\nu}{b^2}\right) \tag{36}$$

$$(M_x)_{(a,0)}=-D\left[(a_{20}+3a_{30})\frac{2}{a^2}+(a_{02}+a_{12})\frac{2\nu}{b}\right] \tag{37}$$

$$(M_x)_{(0,b)}=-D\left[(a_{20}+a_{21})\frac{2}{a^2}+(a_{02}+3a_{03})\frac{2\nu}{b^2}\right] \tag{38}$$

$$(M_x)_{(a,b)}=-D\left[(a_{20}+a_{21}+3a_{30}+3a_{31})\frac{2}{a}+(a_{02}+a_{12}+3a_{03}+3a_{13})\frac{2\nu}{b^2}\right] \tag{39}$$

$$(M_y)_{(0,0)}=-D\left(a_{02}\frac{2}{b^2}+a_{20}\frac{2\nu}{a^2}\right) \tag{40}$$

$$(M_y)_{(a,0)}=-D\left[(a_{02}+a_{12})\frac{2}{b^2}+(a_{20}+3a_{30})\frac{2\nu}{a^2}\right] \tag{41}$$

$$(M_y)_{(0,b)}=-D\left[(a_{02}+3a_{03})\frac{2}{b^2}+(a_{20}+a_{21})\frac{2\nu}{a^2}\right] \tag{42}$$

$$(M_y)_{(a,b)}=-D\left[(a_{02}+a_{12}+3a_{03}+3a_{13})\frac{2}{b^2}+(a_{20'}+a_{21}+3a_{30}+3a_{31})\frac{2\nu}{a^2}\right] \tag{43}$$

$$\begin{aligned}R_{(0,0)}=-2D(1-\nu)\Bigg[&\sum_{m=1}\alpha^2\left(A_m\mathrm{cth}\,\alpha b+B_m\mathrm{cth}\,\alpha b-\frac{C_m}{\mathrm{sh}\alpha b}-\frac{D_m}{\mathrm{sh}\alpha b}\right)\\&+\sum_{n=1}\beta^2\left(E_n\mathrm{cth}\beta a+F_n\mathrm{cth}\beta a-\frac{G_n}{\mathrm{sh}\beta a}-\frac{H_n}{\mathrm{sh}\beta a}\right)\\&-a_{11}\frac{1}{ab}-\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\Bigg]\end{aligned} \tag{44}$$

$$\begin{aligned}R_{(a,0)}=-2D(1-\nu)\Bigg\{&\sum_{m=1}\alpha^2\left(A_m\mathrm{cth}\,\alpha b+B_m\mathrm{cth}\,\alpha b-\frac{C_m}{\mathrm{sh}\alpha b}-\frac{D_m}{\mathrm{sh}\alpha b}\right)\cos m\pi\\&+\sum_{n=1}\beta^2\left[\frac{E_n}{\mathrm{sh}\beta a}+\frac{F_n}{\mathrm{sh}\beta a}-G_n\mathrm{cth}\beta a-H_n(\mathrm{cth}\beta a+\beta a)\right]\\&-(a_{11}+2a_{21}+3a_{31})\frac{1}{ab}-\sum_{m=1}\sum_{n=1}\alpha\beta\cos m\pi\Bigg\}\end{aligned} \tag{45}$$

$$\begin{aligned}R_{(0,b)}=-2D(1-\nu)\Bigg\{&\sum_{m=1}\alpha^2\left[\frac{A_m}{\mathrm{sh}\alpha b}+\frac{B_m}{\mathrm{sh}\alpha b}-C_m\mathrm{cth}\alpha b-D(\mathrm{cth}\alpha b+\alpha b)\right]\\&+\sum_{n=1}\beta^2\left(E_n\mathrm{cth}\beta a+F_n\mathrm{cth}\beta a-\frac{G_n}{\mathrm{sh}\beta a}-\frac{H_n}{\mathrm{sh}\beta a}\right)\cos n\pi\\&(a_{11}+2a_{12}+3a_{13})\frac{1}{ab}-\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos n\pi\Bigg\}\end{aligned} \tag{46}$$

$$R_{(a,b)}=-2D(1-\nu)\Big\{\sum_{m=1}\alpha^2\Big[\frac{A_m}{\mathrm{sh}\alpha b}+\frac{B_m}{\mathrm{sh}\alpha b}-C_m\mathrm{cth}\alpha b-D_m(\mathrm{cth}\alpha b+\alpha b)\Big]\cos m\pi$$

$$+\sum_{n=1}\beta^2\Big[\frac{E_n}{\mathrm{sh}\beta a}+\frac{F_n}{\mathrm{sh}\beta a}-G_n\mathrm{cth}\beta a-H_n(\mathrm{cth}\beta a+\beta a)\Big]\cos n\pi$$

$$-(a_{11}+2a_{21}+2a_{12}+3a_{31}+3a_{13})\frac{1}{ab}-\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos m\pi\cos n\pi\Big\} \quad (47)$$

如果某一边界的挠度曲线为已知条件，则角点的转角条件必须改用角点挠度的二次导数来代替，因此代替角点的转角条件分别有

$$\frac{\partial^2W}{\partial x^2}_{(0,0)}=a_{20}\frac{2}{a^2} \quad (48)$$

$$\frac{\partial^2W}{\partial x^2}_{(a,0)}=(a_{20}+3a_{30})\frac{2}{a^2} \quad (49)$$

$$\frac{\partial^2W}{\partial x^2}_{(0,b)}=(a_{20}+a_{21})\frac{2}{a^2}, \quad (50)$$

$$\frac{\partial^2W}{\partial x^2}_{(a,b)}=(a_{20}+a_{21}+3a_{30}+3a_{31})\frac{2}{a^2} \quad (51)$$

$$\frac{\partial^2W}{\partial y^2}_{(0,0)}=a_{02}\frac{2}{b^2} \quad (52)$$

$$\frac{\partial^2W}{\partial y^2}_{(a,0)}=(a_{02}+a_{12})\frac{2}{b^2} \quad (53)$$

$$\frac{\partial^2W}{\partial y^2}_{(0,b)}=(a_{02}+3a_{03})\frac{2}{b^2} \quad (54)$$

$$\frac{\partial^2W}{\partial y^2}_{(a,b)}=(a_{02}+a_{12}+3a_{03}+3a_{13})\frac{2}{b^2} \quad (55)$$

三、例

以两相邻边固定两相邻边自由的矩形板为例，边界条件和角点条件分别是

$$(W)_{x=0}=\left(\frac{\partial W}{\partial x}\right)_{x=0}=0$$

$$(M_x)_{x=a}=(V_x)_{x=a}=0$$

$$(W)_{y=0}=\left(\frac{\partial W}{\partial y}\right)_{y=0}=0$$

$$(M_y)_{y=b}=(V_y)_{y=b}=0$$

$$W_{(0,0)}=\frac{\partial W}{\partial x}_{(0,0)}=\frac{\partial W}{\partial y}_{(0,0)}=0$$

$$W_{(a,0)}=\frac{\partial W}{\partial y}_{(a,0)}=(M_x)_{(a,0)}=0$$

$$W_{(0,b)}=\frac{\partial W}{\partial x}_{(0,b)}=(M_y)_{(0,b)}=0$$

$$R_{(a,b)}=(M_x)_{(a,b)}=(M_y)_{(a,b)}=0$$

如上所述，四个角点的转角条件

$$\frac{\partial W}{\partial x}_{(0,0)}=\frac{\partial W}{\partial y}_{(0,0)}=\frac{\partial W}{\partial y}_{(a,0)}=\frac{\partial W}{\partial x}_{(0,b)}=0$$

应改用以下的四个条件来代替

$$\frac{\partial^2 W}{\partial x^2}_{(0,0)}=\frac{\partial^2 W}{\partial y^2}_{(0,0)}=\frac{\partial^2 W}{\partial x^2}_{(a,0)}=\frac{\partial^2 W}{\partial y^2}_{(0,b)}=0$$

将等式 (8)，(10)，(12)，(14)，(17)，(19)，(21)，(23)，(24)，(25)，(26)，(37)，(39)，(42)，(43)，(47)，(48)，(49)，(52)，(54) 分别代入以上各式，即令这些等式等于零，便可确定待定常数，为求解方便，首先由等式 (24)，(25)，(26)，(37)，(39)，(42)，(43)，(48)，(49)，(52)，(54) 可以求得：

$$a_{00}=a_{10}=a_{01}=a_{20}=a_{02}=a_{21}=a_{12}=a_{30}=a_{03}=a_{31}=a_{13}=0$$

由等式 (8) 和(10)得

$$A_m=E_n=0$$

由等式(17)和(19)得

$$G_n=F_n\frac{\beta a}{\mathrm{sh}\beta a}-H_n\left(\frac{2}{1-\nu}+\beta a\mathrm{cth}\beta a\right)$$

$$C_m=B_m\frac{\alpha b}{\mathrm{sh}\alpha b}-D_m\left(\frac{2}{1-\nu}+\alpha b\mathrm{cth}\alpha b\right)$$

将以上各式代入等式(12)，(14)，(21)，(23)，(47)化简后得

$$-\sum_{m=1}\frac{4\alpha\beta}{b(\alpha^2+\beta^2)^2}\left[B_m\alpha^2-D_m\frac{(2-\nu)\alpha^2+\beta^2}{1-\nu}\cos n\pi\right]$$

$$-\beta\left[F_n\left(\mathrm{cth}\beta a-\frac{\beta a}{\mathrm{sh}^2\beta a}\right)+\frac{H_n}{\mathrm{sh}\beta a}\left(\frac{1+\nu}{1-\nu}+\beta a\mathrm{cth}\beta a\right)\right]$$

$$-a_{11}\frac{2\cos n\pi}{n\pi a}+\sum_{m=1}A_{mn}\alpha=0 \tag{56}$$

$$-\alpha\left[B_m\left(\mathrm{cth}\alpha b-\frac{\alpha b}{\mathrm{sh}^2\alpha b}\right)+\frac{D_m}{\mathrm{sh}\alpha b}\left(\frac{1+\nu}{1-\nu}+\alpha b\mathrm{cth}\alpha b\right)\right]$$

$$-\sum_{n=1}\frac{4\alpha b}{a(\alpha^2+\beta^2)^2}\left[F_n\beta^2-H_n\frac{(2-\nu)\beta^2+\alpha^2}{1-\nu}\cos m\pi\right]$$

$$-a_{11}\frac{2\cos m\pi}{m\pi b}+\sum_{n=1}A_{mn}\beta=0 \tag{57}$$

$$
\begin{aligned}
&-\sum_{m=1}\frac{4\alpha^3\beta\cos m\pi}{b(\alpha^2+\beta^2)^2}\{B_m[\alpha^2+(2-\nu)\beta^2]-D_m\beta^2\cos n\pi\}\\
&-\beta^2\left\{\frac{F_n}{\operatorname{sh}\beta a}[1+\nu+(1-\nu)\beta a\operatorname{cth}\beta a]-H_n\left[(3+\nu)\operatorname{cth}\beta a+(1-\nu)\frac{\beta a}{\operatorname{sh}^2\beta a}\right]\right\}\\
&+\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2]\cos m\pi=0
\end{aligned}\tag{58}
$$

$$
\begin{aligned}
&-\alpha^3\left\{\frac{B_m}{\operatorname{sh}\alpha b}[1+\nu+(1-\nu)\alpha b\operatorname{cth}\alpha b]-D_m\left[(3+\nu)\operatorname{cth}\alpha b+(1-\nu)\frac{\alpha b}{\operatorname{sh}^2\alpha b}\right]\right\}\\
&-\sum_{n=1}\frac{4\alpha\beta^3\cos n\pi}{a(\alpha^2+\beta^2)^2}\{F_n[(2-\nu)\alpha^2+\beta^2]-H_n\alpha^2\cos m\pi\}\\
&+\sum_{n=1}A_{mn}\beta[(2-\nu)\alpha^2+\beta^2]\cos n\pi=0
\end{aligned}\tag{59}
$$

$$
\begin{aligned}
&\sum_{m=1}\alpha^2\left[\frac{B_m}{\operatorname{sh}\alpha b}(1-\alpha b\operatorname{cth}\alpha b)+D_m\left(\frac{1+\nu}{1-\nu}\operatorname{cth}\alpha b+\frac{\alpha b}{\operatorname{sh}^2\alpha b}\right)\right]\cos m\pi\\
&+\sum_{n=1}\beta^2\left[\frac{F_n}{\operatorname{sh}\beta a}(1-\beta a\operatorname{cth}\beta a)+H_n\left(\frac{1+\nu}{1-\nu}\operatorname{cth}\beta a+\frac{\beta a}{\operatorname{sh}^2\beta a}\right)\right]\cos n\pi\\
&-\frac{a_{11}}{ab}-\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos m\pi\cos n\pi=0
\end{aligned}\tag{60}
$$

设荷载为均布力，此时 q 等于常数，由等式(3)得

$$
A_{mn}=\frac{4q(1-\cos m\pi)(1-\cos n\pi)}{Dab\alpha\beta(\alpha^2+\beta^2)^2}
$$

将上式代入等式(56)至(60)得与 A_{mn} 有关的各项为：

$$
\left.
\begin{aligned}
&\sum_{m=1}A_{mn}\alpha=\frac{4q(1-\cos n\pi)}{Dab\beta}\sum_{m=1}\frac{1-\cos m\pi}{(\alpha^2+\beta^2)^2}\\
&\sum_{n=1}A_{mn}\beta=\frac{4q(1-\cos m\pi)}{Dab\alpha}\sum_{n=1}\frac{1-\cos n\pi}{(\alpha^2+\beta^2)^2}\\
&\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2]\cos m\pi=\\
&\quad-\frac{4q(1-\cos n\pi)}{Dab\beta}\sum_{m=1}\frac{1-\cos m\pi}{(\alpha^2+\beta^2)^2}[\alpha^2+(2-\nu)\beta^2]\\
&\sum_{n=1}A_{mn}\beta[(2-\nu)\alpha^2+\beta^2]\cos n\pi=\\
&\quad-\frac{4q(1-\cos m\pi)}{Dab\alpha}\sum_{n=1}\frac{1-\cos n\pi}{(\alpha^2+\beta^2)^2}[(2-\nu)\alpha^2+\beta^2]\\
&\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos m\pi\cos n\pi=\sum_{n=1}\frac{4q(1-\cos n\pi)}{Dab}\sum_{m=1}\frac{1-\cos m\pi}{(\alpha^2+\beta^2)^2}
\end{aligned}
\right\}\tag{61}
$$

设荷载为集中力，作用点在(ξ,η)，由等式(3)得

$$
A_{mn}=\frac{4P\sin\alpha\xi\sin\beta\eta}{Dab(\alpha^2+\beta^2)^2}
$$

将上式代入等式(56)至(60)得

$$
\left.\begin{aligned}
&\sum_{m=1} A_{mn}\alpha=\frac{4P\sin\beta\eta}{Dab}\sum_{m=1}\frac{\alpha\sin\alpha\xi}{(\alpha^2+\beta^2)^2}\\
&\sum_{n=1} A_{mn}\beta=\frac{4P\sin\alpha\xi}{Dab}\sum_{n=1}\frac{\beta\sin\beta\eta}{(\alpha^2+\beta^2)^2}\\
&\sum_{m=1} A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2)\cos m\pi\\
&=\frac{4P\sin\beta\eta}{Dab}\sum_{m=1}\frac{\alpha\text{nis}\alpha\xi}{(\alpha^2+\beta^2)^2}[\alpha^2+(2-\nu)\beta^2]\cos m\pi\\
&\sum_{n=1} A_{mn}\beta[(2-\nu)\alpha^2+\beta^2]\cos n\pi\\
&=\frac{4P\text{nis}\alpha\xi}{Dab}\sum_{n=1}\frac{\beta\text{nis}\beta\eta}{(\alpha^2+\beta^2)^2}[(2-\nu)\alpha^2+\beta^2]\cos n\pi\\
&\sum_{m=1}\sum_{n=1} A_{mn}\alpha\beta\cos m\pi\cos n\pi\\
&=\sum_{n=1}\frac{4P\beta\text{nis}\beta\eta}{Dab}\cos n\pi\sum_{m=1}\frac{\alpha\sin\alpha\xi}{(\alpha^2+\beta^2)^2}\cos m\pi
\end{aligned}\right\}\quad(62)
$$

等式(61)和(62)中存在着许多求和项，也可以通过数学演算进行求和，应用函数的余弦级数展开式：

$$\text{ch}\beta x=\frac{\text{sh}\beta a}{\beta a}+\sum_{m=1}\frac{2\beta\text{sh}\beta a\cos m\pi}{a(\alpha^2+\beta^2)}\cos\alpha x$$

$$\text{sh}\beta x=\frac{\text{ch}\beta a-1}{\beta a}+\sum_{m=1}\frac{2\beta(\text{ch}\beta a\cos m\pi-1)}{a(\alpha^2+\beta^2)}\cos\alpha x$$

$$\beta x\text{ch}\beta x=\text{sh}\beta a-\frac{\text{ch}\beta a-1}{\beta a}+\sum_{m=1}\frac{2\beta}{a(\alpha^2+\beta^2)}\left[\beta a\text{sh}\beta a\cos m\pi+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}(\text{ch}\beta a\cos m\pi-1)\right]\cos\alpha x$$

$$\beta x\text{sh}\beta x=\text{ch}\beta a-\frac{\text{sh}\beta a}{\beta a}+\sum_{m=1}\frac{2\beta}{a(\alpha^2+\beta^2)}\left[\beta a\text{ch}\beta a+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\text{sh}\beta a\right]\cos m\pi\cos\alpha x$$

以及

$$\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}=1-\frac{2\beta^2}{\alpha^2+\beta^2}=\frac{2\alpha^2}{\alpha^2+\beta^2}-1$$

可以求得

$$\sum_{m=1}\frac{\cos\alpha x}{(\alpha^2+\beta^2)^2}=-\frac{a}{4\beta^3}\left[\frac{2}{\beta a}-\left(\text{cth}\beta a+\frac{\beta a}{\text{sh}^2\beta a}\right)\text{ch}\beta x+\text{cth}\beta a\cdot\beta x\text{sh}\beta x+\text{sh}\beta x-\beta x\text{ch}\beta x\right]$$

$$\sum_{m=1}\frac{\alpha^2\cos\alpha x}{(\alpha^2+\beta^2)^2}=\frac{a}{4\beta}\left[\left(\mathrm{cth}\beta a-\frac{\beta a}{\mathrm{sh}^2\beta a}\right)\mathrm{ch}\beta x+\mathrm{cth}\beta a\cdot\beta x\mathrm{sh}\beta x-\mathrm{sh}\beta x-\beta x\mathrm{ch}\beta x\right]$$

在以上二式中分别令$x=0$和$x=a$，然后将所得结果代入等式(61)的第一、三、五式，得

$$\left.\begin{aligned}&\sum_{m=1}A_{mn}\alpha=\frac{q(1-\cos n\pi)}{Db\beta^4}\left(\mathrm{cth}\beta a-\frac{1}{\mathrm{sh}\beta a}\right)\left(1-\frac{\beta a}{\mathrm{sh}\beta a}\right)\\&\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2]\cos m\pi=-\frac{q(1-\cos n\pi)}{Db\beta^2}\\&\quad\cdot\left(\mathrm{cth}\beta a-\frac{1}{\mathrm{sh}\beta a}\right)\left[3-\nu-(1-\nu)\frac{\beta a}{\mathrm{sh}\beta a}\right]\\&\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos m\pi\cos n\pi=\sum_{n=1}\frac{q(1-\cos n\pi)}{Db\beta^3}\\&\quad\cdot\left(\mathrm{cth}\beta a-\frac{1}{\mathrm{sh}\beta a}\right)\left(1-\frac{\beta a}{\mathrm{sh}\beta a}\right)\end{aligned}\right\}\qquad(63)$$

同样有

$$\left.\begin{aligned}&\sum_{n=1}A_{mn}\beta=\frac{q(1-\cos m\pi)}{Da\alpha^4}\left(\mathrm{cth}\alpha b-\frac{1}{\mathrm{sh}\alpha b}\right)\left(1-\frac{\alpha b}{\mathrm{sh}\alpha b}\right)\\&\sum_{n=1}A_{mn}\beta[(2-\nu)\alpha^2+\beta^2]\cos n\pi=-\frac{q(1-\ \ \mathrm{os}m\pi)}{Da\alpha^2}\\&\quad\cdot\left(\mathrm{cth}\alpha b-\frac{1}{\mathrm{sh}\alpha b}\right)\left[3-\nu-(1-\nu)\frac{\alpha b}{\mathrm{sh}\alpha b}\right]\end{aligned}\right\}\qquad(64)$$

同样应用函数的正弦级数展开式可以求得

$$\sum_{m=1}\frac{\alpha\sin\alpha x}{(\alpha^2+\beta^2)^2}=-\frac{a}{4\beta^2}\left(\beta x\mathrm{sh}\beta x+\frac{\beta a}{\mathrm{sh}^2\beta a}\mathrm{sh}\beta x-\mathrm{cth}\beta a\cdot\beta x\mathrm{ch}\beta x\right)$$

$$\sum_{m=1}\frac{\alpha\sin\alpha x}{(\alpha^2+\beta^2)^2}\cos m\pi=-\frac{a}{4\beta^2\mathrm{sh}\beta a}(\beta a\mathrm{cth}\beta a\mathrm{sh}\beta x-\beta x\mathrm{ch}\beta x)$$

$$\sum_{m=1}\frac{\alpha^3\sin\alpha x}{(\alpha^2+\beta^2)^2}\cos m\pi=-\frac{a}{4\mathrm{sh}\beta a}[(2-\beta a\mathrm{cth}\beta a)\mathrm{sh}\beta x+\beta x\mathrm{ch}\beta x]$$

在以上三式中令$x=\xi$，然后将所得结果代入等式(62)的第一、三、五式，得

$$\left.\begin{aligned}&\sum_{m=1}A_{m}\alpha=\frac{P\sin\beta\eta\mathrm{sh}\beta\xi}{Db\beta^2}\left[\beta\xi(\mathrm{cth}\beta\xi\mathrm{cth}\,\beta a-1)-\frac{\beta a}{\mathrm{sh}^2\beta a}\right]\\&\sum_{m=1}A_{mn}\alpha[\alpha^2+(2-\nu)\beta^2)]\cos m\pi=-\frac{P\sin\beta\eta\mathrm{sh}\beta\xi}{Db\mathrm{sh}\beta a}\\&\quad\cdot[2+(1-\nu)(\beta a\mathrm{cth}\beta a-\beta\xi\mathrm{cth}\beta\xi)]\\&\sum_{m=1}\sum_{n=1}A_{mn}\alpha\beta\cos m\pi\cos n\pi=\sum_{n=1}\frac{P\sin\beta\eta\mathrm{sh}\beta\xi}{Db\beta\mathrm{sh}\beta a}\\&\quad\cdot(\beta\xi\mathrm{cth}\beta\xi-\beta a\mathrm{cth}\beta a)\cos n\pi\end{aligned}\right\}\qquad(55)$$

同样有

$$\left.\begin{aligned}&\sum_{n=1} A_{nn}\beta=\frac{P\sin\alpha\xi\,\mathrm{sh}\,\alpha\eta}{D\alpha a^2}\left[\alpha\eta(\mathrm{cth}\,\alpha b\,\mathrm{cth}\,\alpha\eta-1)-\frac{\alpha b}{\mathrm{sh}^2\alpha b}\right]\\&\sum_{n=1} A_{mn}\beta\left[(2-\nu)\alpha^2+\beta^2)\cos n\pi\right]=-\frac{P\sin\alpha\xi\,\mathrm{sh}\,\alpha\eta}{D\alpha\,\mathrm{sh}\,\alpha b}\\&\quad\cdot\left[2+(1-\nu)(\alpha b\,\mathrm{cth}\,\alpha b-\alpha\eta\,\mathrm{cth}\,\alpha\eta)\right]\end{aligned}\right\}\qquad(66)$$

等式(63)至(66)和用单正弦级数解法求得的承受同样荷载的四边简支矩形板的结果是相同的，为计算简单起见，取$a=b$，在均布力时将有 $F_n=B_n$，$H_n=D_n$，根据等式(56)，(58)，(60)幷分别应用等式(61)和(63)，取 $\nu=0.3$，m和n分别由1取至2，4，10，20时算得 a_{11}（即自由角点的挠度）的结果见表1，表中单位为 $10^{-6}qa^4/D$，表中当m和n取到50时的结果即系文献[4]计算的结果。

表 1

	2	4	10	20	50
(61)	36056	39203	41358	42171	
(62)	56977	50341	45893	44442	43678

仍然取 $a=b$，当集中力作用在(0,0)和(a,b)联线的任一点上将有 $\xi=\eta$，$F_n=B_n$，$H_n=D_n$，根据等式(56)，(58)，(60)幷分别应用等式(61)和(65)，取$\nu=0.3$、ξ/a分别等于0.25，0.5，0.75，m和n分别由1取至4，8，12，16，20时算得a_{11}的结果见表2，表中单位为$10^{-6}P/D$，当集中力作用在自由角点，取m和n由1至50时$a_{11}=297867\times10^{-6}P/D$[4]。

		4	8	12	16	20
0.25	(62)	5555	608	2576	1171	2165
	(65)	2256	1828	1750	1723	1712
0.50	(62)	20169	24854	26621	27538	28103
	(65)	37021	33147	32060	31574	31308
0.75	(62)	212532	92441	150718	106138	138585
	(65)	135875	127300	125108	124083	123510

从两个表中可以看出，单行项（用等式(61)，(62)计算时）由小变大，双行项（用等式(63)，(65)计算时）由大变小幷逐渐接近精确值，故同时采用两种特解计算可以求得精确值的上下限。在表2中单行项差值较大，这是因为等式(62)中级数的收斂性差，而且由式中的 $\sin\alpha\xi$ 可以看出，当$\xi/a=0.5$时，函数的周期是$m=4$，故该行各项均由由小变大；当 $\xi/a=0.25$ 和 0.75 时，函数周期是$m=8$，故表中由8增至16确信是由小变大幷逐渐接近精确值，而其他各值是不可引用的。

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A General Solution for Solving Elastic Bending Problem of Rectangular Thin Plates

Huang Yan

Abstract

This paper gives a general solution for solving elastic bending problem of rectangular thin plates, and then the intergral constants are determined by means of boundary conditions. This method is simplerand easier than superposition method.