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矩形薄板自由振动的解析解法

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摘　要　本文建立了矩形薄板横向自由振动振型函数微分方程的一般解，可以求解任意边界条件的矩形板的振动问题。以一组对边固定一组对边自由的矩形板为例求解了板的基本频率。

关键词　自由振动　基本频率

1.　微分方程的解

如图1所示，矩形薄板弹性横向自由振动振型函数的微分方程为[1]

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-\gamma^4 w=0 \qquad (1.1)$$

式中

$$\gamma^4=\frac{\rho\omega^2}{D} \qquad (1.2)$$

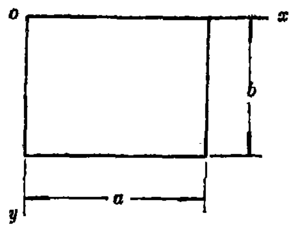

图 1

ω 为板的固有频率、ρ 为单位面积质量、D 为弯曲刚度。采用分离变量法，设 $w=X_{(x)}Y_{(y)}$，代入(1.1)式可得

$$Y\frac{\partial^4 X}{\partial x^4}+2\frac{\partial^2 X}{\partial x^2}\frac{\partial^2 Y}{\partial y^2}+X\frac{\partial^4 Y}{\partial y^4}-\gamma^4 XY=0 \qquad (1.3)$$

将上式除以 XY，然后对 y 微分一次得

$$\frac{X''}{X}=-\frac{1}{2}\left(\frac{Y^{\mathrm{IV}}}{Y}\right)'\Big/\left(\frac{Y''}{Y}\right)'$$

上式两边必为一常数，设为 $-\alpha^2$，故有

$$X''+\alpha^2 X=0 \qquad (1.4)$$

上式的解分为两种：(1)当 α 为零时可得

$$X=A_1+A_2 x \qquad (1.5)$$

将上式代入(1.3) 式可得

$$Y^{\mathrm{IV}}-\gamma^4 Y=0$$

上式特征方程的根为 $\pm\gamma$ 和 $\pm i\gamma$，即

$$Y=B_1\mathrm{sh}\gamma y+B_2\mathrm{ch}\gamma y+B_3\sin\gamma y+B_4\cos\gamma y \qquad (1.6)$$

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(2) 当α不为零时，由(1.4)式可得

$$X = C_1 \sin\alpha x + C_2 \cos\alpha x \tag{1.7}$$

将上式代入(1.3) 式可得

$$Y^{\text{IV}} - 2\alpha^2 Y'' + (\alpha^4 - \gamma^4) Y = 0$$

上式特征方程的根又分两种情形：一是当$\gamma > \alpha$时为$\pm\alpha_1$和$\pm i\alpha_2$

$$\alpha_1 = \sqrt{\gamma^2 + \alpha^2}, \quad \alpha_2 = \sqrt{\gamma^2 - \alpha^2} \tag{1.8}$$

$$Y = D_1 \mathrm{sh}\alpha_1 y + D_2 \mathrm{ch}\alpha_1 y + D_3 \sin\alpha_2 y + D_4 \cos\alpha_2 y \tag{1.9}$$

另一是当$\gamma < \alpha$时为$\pm\alpha_1$和$\pm\alpha_3$

$$\alpha_3 = \sqrt{\alpha^2 - \gamma^2} \tag{1.10}$$

$$Y = D_1 \mathrm{sh}\alpha_1 y + D_2 \mathrm{ch}\alpha_1 y + D_3 \mathrm{sh}\alpha_3 y + D_4 \mathrm{ch}\alpha_3 y \tag{1.11}$$

如将(1.3)式除以XY，然后对x微分一次，则同样可得相似的另一类解，适当选取各种解的组合可以求解各种不同边界矩形板的自由振动问题。

2. 一般解的建立

满足四边以及四角为任意边界条件的矩形板的振动问题本文取为：

$$\begin{aligned}
w = {} & \sum_m \left[A_m \frac{\mathrm{sh}\alpha_1(b-y)}{\mathrm{sh}\alpha_1 b} + B_m \frac{\mathrm{sh}\alpha_1 y}{\mathrm{sh}\alpha_1 b} \right] \sin\alpha x \\
& + \sum_{m<M} \left[C_m \frac{\sin\alpha_2(b-y)}{\sin\alpha_2 b} + D_m \frac{\sin\alpha_2 y}{\sin\alpha_2 b} \right] \sin\alpha x \\
& + \sum_{m>M} \left[C_m \frac{\mathrm{sh}\alpha_3(b-y)}{\mathrm{sh}\alpha_3 b} + D_m \frac{\mathrm{sh}\alpha_3 y}{\mathrm{sh}\alpha_3 b} \right] \sin\alpha x \\
& + \sum_n \left[E_n \frac{\mathrm{sh}\beta_1(a-x)}{\mathrm{sh}\beta_1 a} + F_n \frac{\mathrm{sh}\beta_1 x}{\mathrm{sh}\beta_1 a} \right] \sin\beta y \\
& + \sum_{n<N} \left[G_n \frac{\sin\beta_2(a-x)}{\sin\beta_2 a} + H_n \frac{\sin\beta_2 x}{\sin\beta_2 a} \right] \sin\beta y \\
& + \sum_{n>N} \left[G_n \frac{\mathrm{sh}\beta_3(a-x)}{\mathrm{sh}\beta_3 a} + H_n \frac{\mathrm{sh}\beta_3 x}{\mathrm{sh}\beta_3 a} \right] \sin\beta y \\
& + \left[A \frac{\mathrm{sh}\gamma(b-y)}{\mathrm{sh}\gamma b} + B \frac{\mathrm{sh}\gamma y}{\mathrm{sh}\gamma b} \right] \frac{a-x}{a} + \left[C \frac{\mathrm{sh}\gamma(b-y)}{\mathrm{sh}\gamma b} + D \frac{\mathrm{sh}\gamma y}{\mathrm{sh}\gamma b} \right] \frac{x}{a} \\
& + \left[E \frac{\mathrm{sh}\gamma(a-x)}{\mathrm{sh}\gamma a} + F \frac{\mathrm{sh}\gamma x}{\mathrm{sh}\gamma a} \right] \frac{b-y}{b} + \left[G \frac{\mathrm{sh}\gamma(a-x)}{\mathrm{sh}\gamma a} + H \frac{\mathrm{sh}\gamma x}{\mathrm{sh}\gamma a} \right] \frac{y}{b} \\
& + I \left[\frac{\sin\gamma(b-y)}{\sin\gamma b} \frac{a-x}{a} + \frac{\sin\gamma(a-x)}{\sin\gamma a} \frac{b-y}{b} \right] \\
& + J \left[\frac{\sin\gamma(b-y)}{\sin\gamma b} \frac{x}{a} + \frac{\sin\gamma x}{\sin\gamma a} \frac{b-y}{b} \right] + K \left[\frac{\sin\gamma y}{\sin\gamma b} \frac{a-x}{a} + \frac{\sin\gamma(a-x)}{\sin\gamma a} \frac{y}{b} \right] \\
& + L \left[\frac{\sin\gamma y}{\sin\gamma b} \frac{x}{a} + \frac{\sin\gamma x}{\sin\gamma a} \frac{y}{b} \right]
\end{aligned} \tag{2.1}$$

式中 $\alpha=\frac{m\pi}{a}$， $m=1,2,3\cdots\cdots$ (2.2)

$$\beta=\frac{n\pi}{b}, \quad n=1,2,3\cdots\cdots \tag{2.3}$$

$$\beta_1=\sqrt{\gamma^2+\beta^2},\ \beta_2=\sqrt{\gamma^2-\beta^2},\ \beta_3=\sqrt{\beta^2-\gamma^2} \tag{2.4}$$

$$M=\frac{\gamma a}{\pi},\ N=\frac{\gamma b}{\pi} \tag{2.5}$$

(2.1)式含有$4m+4n+12$个积分常数。(2.1)式的第一部分能满足$y=0$和$y=b$两个边为任意的问题；第二部分能满足$x=0$和$x=a$两个边为任意的问题；第三部分能满足四个角为任意的问题。由于每个边有挠度或等效剪力，斜度或弯矩二个边界条件，将每个边界条件所建立的方程式中非正弦级数均展成正弦级数，则利用正弦级数的正交性可得$4m$和$4n$个方程式，加上每个角有挠度或反力，角的两边的斜度或弯矩三个条件，总共可以求解$4m+4n+12$个未知数。(2.1)式需展成的正弦级数如下：

$$\frac{\mathrm{sh}\alpha'(b-y)}{\mathrm{sh}\alpha' b}=\sum_n\frac{2n\pi}{(n\pi)^2+(\alpha' b)^2}\sin\beta y \tag{2.6}$$

$$\frac{\mathrm{sh}\alpha' y}{\mathrm{sh}\alpha' b}=-\sum_n\frac{2n\pi\cos n\pi}{(n\pi)^2+(\alpha' b)^2}\sin\beta y \tag{2.7}$$

$$\frac{\sin\alpha'(b-y)}{\sin\alpha' b}=\sum_n\frac{2n\pi}{(n\pi)^2-(\alpha' b)^2}\sin\beta y \tag{2.8}$$

$$\frac{\sin\alpha' y}{\sin\alpha' b}=-\sum_n\frac{2n\pi\cos n\pi}{(n\pi)^2-(\alpha' b)^2}\sin\beta y \tag{2.9}$$

$$\frac{b-y}{b}=\sum_n\frac{2}{n\pi}\sin\beta y \tag{2.10}$$

$$\frac{y}{b}=-\sum_n\frac{2\cos n\pi}{n\pi}\sin\beta y \tag{2.11}$$

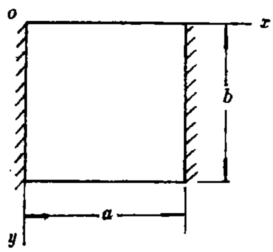

图 2

3. 算 例

如图 2 所示，以一组对边固定一组对边自由为例，边界条件是：

$$w_{x=0}=0,\ w_{x=a}=0 \tag{3.1}$$

$$\left(\frac{\partial w}{\partial x}\right)_{x=0}=0,\ \left(\frac{\partial w}{\partial x}\right)_{x=a}=0 \tag{3.2}$$

$$(M_y)_{y=0}=0,\quad (M_y)_{y=b}=0 \tag{3.3}$$

$$(V_y)_{y=0}=0,\quad (V_y)_{y=b}=0 \tag{3.4}$$

式中 $M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$， $V_y=-D\left[\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial y\partial x^2}\right]$

角点条件是：

$$w_{(0,0)}=0,\ w_{(a,0)}=0,\ w_{(0,b)}=0,\ w_{(a,b)}=0 \tag{3.5}$$

$$\frac{\partial w}{\partial y}_{(0,0)}=0,\ \frac{\partial w}{\partial y}_{(a,0)}=0,\ \frac{\partial w}{\partial y}_{(0,b)}=0,\ \frac{\partial w}{\partial y}_{(a,b)}=0 \tag{3.6}$$

$$M_{x(0,0)}=0,\ M_{x(a,0)}=0,\ M_{x(0,b)}=0,\ M_{x(a,b)}=0 \tag{3.7}$$

为使计算简单起见，代替(3.6)式，改取[2]

$$\frac{\partial^2 w}{\partial y^2}_{(0,0)}=0,\ \frac{\partial^2 w}{\partial y^2}_{(a,0)}=,\ \frac{\partial^2 w}{\partial y^2}_{(0,b)}=0,\ \frac{\partial^2 w}{\partial y^2}_{(a,b)}=0 \tag{3.8}$$

由于所研究的板具有$x=a/2$和$y=b/2$两条对称轴，板的振型将是对称的或是反对称的。利用对称和反对称条件可使求解自由振动的问题大大简化。对$x=a/2$为对称或反对称情形应有

$$(M_x)_{x=0}=\pm(M_x)_{x=a} \tag{3.9}$$

正负号同时书写时，上号为对称，下号为反对称。对$y=b/2$为对称或反对称情形应有

$$w_{y=0}=[\pm]w_{y=b} \tag{3.10}$$

上式正负号两边再加一方括号是为了区别$x=a/2$为对称或反对称情形。将(2.1) 式代入以上各式，首先由(3.8)式(3.7)式，(3.5)式可得

$$A=I,\ C=J,\ B=K,\ D=L$$
$$E=I,\ F=J,\ G=K,\ H=L$$
$$I=J=K=L=0$$

由(3.1)式和(3.3)式并应用到(1.8)和(1.10)式以及利用正弦级数的正交性可得

$$E_n=-G_n,\ F_n=-H_n$$

$$C_m=\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}A_m,\ D_m=\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}B_m$$

再由(3.9)和(3.10)式并应用到(2.4)式可得

$$E_n=\pm F_n;\ A_m=[\pm]B_m$$

应用以上各式，最后由(3.2)的第一式和(3.4)的第一式并应用到(2.6)至(2.9)式可得

$$\sum_m A_m 2n\pi\alpha\left[\frac{1[\mp]\cos n\pi}{(n\pi)^2+(\alpha_1 b)^2}+\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}\frac{1[\mp]\cos n\pi}{(n\pi)^2+(\alpha_3 b)^2}\right]$$

$$-E_n\left\{\beta_1\left(\operatorname{cth}\beta_1 a\mp\frac{1}{\operatorname{sh}\beta_1 a}\right)-\begin{bmatrix}\beta_2\left(\operatorname{ctg}\beta_2 a\mp\frac{1}{\sin\beta_2 a}\right),\ \text{当}n<N\\ \beta_3\left(\operatorname{cth}\beta_3 a\mp\frac{1}{\operatorname{sh}\beta_3 a}\right),\ \text{当}n>N\end{bmatrix}\right\}=0$$

$$A_m\left\{[\gamma^2-(1-\nu)\alpha^2]\,\alpha_1\left(\operatorname{cth}\alpha_1 b[\mp]\frac{1}{\operatorname{sh}\alpha_1 b}\right)\right.$$

$$-\frac{[\gamma^2+(1-\nu)\alpha^2]^2}{\gamma^2-(1-\nu)\alpha^2}\begin{bmatrix}\alpha_2(\mathrm{ctg}\alpha_2 b[\mp]\dfrac{1}{\sin\alpha_2 b}),\text{当} m<M\\ \alpha_3(\mathrm{cth}\alpha_3 b[\mp]\dfrac{1}{\mathrm{sh}\alpha_3 b}),\text{当} m>M\end{bmatrix}\Bigg\}$$

$$-\sum_n E_n 2m\pi\beta(1\mp\cos m\pi)\left[\frac{(2-\nu)\gamma^2+(1-\nu)\beta^2}{(n\pi)^2+(\beta_1 a)^2}+\frac{(2-\nu)\gamma^2-(1-\nu)\beta^2}{(n\pi)^2+(\beta_3 a)^2}\right]=0$$

正方形时，令$a=b$，并应用到(1.8)式，(1.10)式和(2.4)式，则以上二式简化为

$$\sum_m A_m 2mn(1[\mp]\cos n\pi)\left[\frac{1}{m^2+n^2+M^2}+\frac{1}{m^2+n^2-M^2}\frac{M^2+(1-\nu)m^2}{M^2-(1-\nu)m^2}\right]$$

$$-E_n\left[n_1\pi\left(\mathrm{cth}n_1\pi\mp\frac{1}{\mathrm{sh}n_1\pi}\right)-\begin{Bmatrix}n_2\pi\left(\mathrm{ctg}n_2\pi\mp\dfrac{1}{\sin n_2\pi}\right),n<M\\ n_3\pi\left(\mathrm{cth}n_3\pi\mp\dfrac{1}{\mathrm{sh}n_3\pi}\right),n>M\end{Bmatrix}\right]=0 \qquad (3.11)$$

$$\sum_m\Bigg\{m_1\pi[M^2-(1-\nu)m^2]\left(\mathrm{cth}m_1\pi[\mp]\frac{1}{\mathrm{sh}m_1\pi}\right)$$

$$-\frac{[M^2+(1-\nu)m^2]^2}{M^2-(1-\nu)m^2}\begin{Bmatrix}m_2\pi\left(\mathrm{ctg}m_2\pi[\mp]\dfrac{1}{\sin m_2\pi}\right),\text{当} m<M\\ m_3\pi\left(\mathrm{cth}m_3\pi[\mp]\dfrac{1}{\mathrm{sh}m_3\pi}\right),\text{当} m>M\end{Bmatrix}\Bigg\}$$

$$-\sum_n E_n 2mn(1[\mp]\cos m\pi)\left[\frac{(2-\nu)M^2+(1-\nu)n^2}{m^2+n^2+M^2}+\frac{(2-\nu)M^2-(1-\nu)n^2}{m^2+n^2-M^2}\right]=0$$

(3.12)

式中 $m_1=\sqrt{M^2+m^2}$，$m_2=\sqrt{M^2-m^2}$，$m_3=\sqrt{m^2-M^2}$

$n_1=\sqrt{M^2+n^2}$，$n_2=\sqrt{M^2-n^2}$，$n_3=\sqrt{n^2-M^2}$

令(3.11)式和(3.12)式的系数行列式等于零，可以求得确定固有频率的M值，采用试算法，令$\nu=0.333$，m和n各取6项算得对$x=a/2$和$y=a/2$分别为对称和反对称共四种情形的前四个M值，见表1至表4.

文献[1]的M值是取$\nu=0.333$，本文的M值略偏小一些，这是因为本文取m和n的项数偏少。当m和n各取10项则表1和表2的第一基频分别为1.4958和2.5911.

表1 对$x=a/2$和$y=a/2$均为对称的M值

	最低的四个频率的M值			
	1	2	3	4
本文	1.4921	2.0741	3.4633	3.7051
文献[1]	1.4967	2.0941	3.4846	3.7214

表2 对$x=a/2$和$y=a/2$均为反对称的M值

M值	1	2	3	4
本文	2.5825	3.5145	4.5174	5.7405
文献[1]	2.6016	3.5417	4.5562	5.8302

表 3　对$x=a/2$为对称$y=a/2$为反对称的M值

M值	1	2	3	4
本文	1.6208	2.8265	3.5537	4.3189
文献[1]	1.6289	2.8385	3.5753	4.3556

表 4　对$x=a/2$为反对称$y=a/2$为对称的M值

M值	1	2	3	4
本文	2.4724	2.9350	4.2233	4.4552
文献[1]	2.4869	2.9697	4.2658	4.4831

文献[1]的M值是取$\gamma=0.333$，本文的M值略偏小一些，这是因为本文取m和n的项数偏少。当m和n各取10项，则表1和表2的第一基频分别为1.4958和2.5911.

文献[1]是用各种解析解的迭加法求得的，求解过程比较复杂，本文的理论分析具有一般性，推理简单，求解容易，便于应用。

本文的计算得到研究生张忠同志的帮助，深表感谢。

参考文献

[1] D.J.Gorman.Free Vibration Analysis of Rectangular Plates.Elsevier,1982

[2] 黄炎．矩形薄板弹性弯曲问题的一般解析解．国防科技大学学报，1983年,第3期

The Damage Analysis for Glass/Epoxy Composites

Yang Guangsong Zhou Minji

Abstract

In this paper, we successfully took the soft x-ray radiographs that distinctly. showed the internal damage such as microcracks in glass/epoxy composites The failure mechanisms and damage evolution law could be effectively obtained from these radiographs co-operating with AE results. Based upon the fact that the orientation of microcracks in fiber composites under loading is invariably in parallell with fiber, we define the damage variables correspond to mode Ⅰ and Ⅱ in macroscopic fracture mechanics, and obtain the constitutive equations for the damaged lamina and laminate. The calculated results coincide closely with the experimental results.

Key words Damage in composite, Micracracks soft x-ray, AE technique

Analytical Method for Soluving Free Vibration of Rectangular Plates

Huang Yan

Abstract

A general solution of differential equation for lateral displacement function in free vibration of rectangular thin plates is established in this paper. It can be used to solve vibration problem of rectangular plate with arbitrary boundaries. For example, the fundamental frequency of a plate with two opposite edges fixed and the other two edges free is solved.

Key words Free vibration Fundamental Frequency