

## 梁的横向振动频率方程的一般表达式

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**摘要** 本文推导出梁的横向振动频率方程的一般表达式。十七种在各种边界条件下的梁的横向振动频率方程均是它的特殊情况。

**关键词**：梁，振动，结构振动，频率方程式。

## 1. 引言

梁是工程中最常用的构件之一。汽轮机中的转轴，工作叶片和导向叶片的弯曲振动也可以简化为梁来考虑。关于梁的弯曲自由振动的频率方程在一般振动理论书中均有叙述<sup>[1]</sup>。但都是根据每一具体的边界条件进行各别推导。文[2]在梁的两端引入不同的弹簧刚度系数 $K_1, K_2$ （产生每单位位移所需的力）或不同的弹性固持座的刚度系数 $\bar{K}_1, \bar{K}_2$ （产生每单位转角所需的力偶值），把具有不同边界条件的梁归为三类情况：（1）两端具有不同弹簧支承的梁（图1）；（2）两端具有不同弹性固持座的梁（图2）；（3）一端弹性固持座，一端弹簧支承梁（图3）。分别推出上述三类具有不同弹性支承梁的横向振动频率方程式。从而可以演化出十六种在各种边界条件下梁的横向振动频率方程。本文采用文[2]的思路，把上述三类情况归并为一类，推导出梁的横向振动频率方程的一般表达式。而其它17种在各种边界条件下的梁的横向振动频率方程均是它的特殊情况。

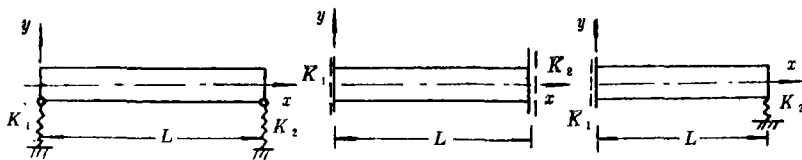


图 1

图 2

图 3

梁弯曲振动的振型函数可写为：<sup>[3]</sup>

$$y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \operatorname{sh} \lambda x + C_4 \operatorname{ch} \lambda x \quad (1.1)$$

式中

$$\lambda^4 = \frac{\rho F p^2}{EI} \quad (1.2)$$

$EI$ 为梁的弯曲刚度， $\rho$ 为梁材密度， $F$ 为梁的横截面积， $p$ 为梁弯曲的固有圆频率。

## 2. 梁的横向振动频率方程的一般表达式

如图4所示。梁两端的弹簧刚度系数和弹性固持座的刚度系数分别为 $K_1, \bar{K}_1$ 和 $K_2, \bar{K}_2$ 。

端点条件为:

$$\left. \begin{aligned} x=0, \quad y(0) &= -\frac{EI}{K_1}y''(0), \quad y''(0) = -\frac{\bar{K}_1}{EI}y'(0) \\ x=L, \quad y(L) &= -\frac{EI}{K_2}y''(L), \quad y''(L) = \frac{\bar{K}_2}{EI}y'(L) \end{aligned} \right\} \quad (2.1)$$

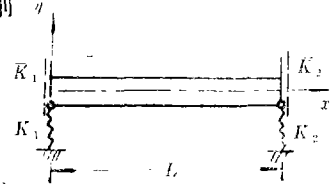


图4

将(1.1)式及其一阶、二阶、三阶导数式代入(2.1)式, 整理后得:

$$\begin{aligned} EI\lambda^3 C_1 - K_1 C_2 - EI\lambda^3 C_3 - K_1 C_4 &= 0 \\ \bar{K}_1 C_1 - EI\lambda C_2 + \bar{K}_1 C_3 + EI\lambda C_4 &= 0 \\ (K_2 \sin \lambda L + EI\lambda^3 \cos \lambda L) C_1 + (K_2 \cos \lambda L - EI\lambda^3 \sin \lambda L) C_2 + \\ (K_2 \operatorname{sh} \lambda L - EI\lambda^3 \operatorname{ch} \lambda L) C_3 + (K_2 \operatorname{ch} \lambda L - EI\lambda^3 \operatorname{sh} \lambda L) C_4 &= 0 \\ (\bar{K}_2 \cos \lambda L + EI\lambda \sin \lambda L) C_1 + (-\bar{K}_2 \sin \lambda L + EI\lambda \cos \lambda L) C_2 + \\ (\bar{K}_2 \operatorname{ch} \lambda L - EI\lambda \operatorname{sh} \lambda L) C_3 + (\bar{K}_2 \operatorname{sh} \lambda L - EI\lambda \operatorname{ch} \lambda L) C_4 &= 0 \end{aligned} \quad (2.2)$$

这是四个关于 $C_1, C_2, C_3$ 和 $C_4$ 的代数联立方程, 要使(2.2)式中的 $C_1, C_2, C_3$ 和 $C_4$ 有非零解, 系数行列式必为零。展开简化后得梁的横向振动频率方程的一般表达式为:

$$\begin{aligned} [E^2 I^2 \lambda^6 \bar{K}_1 \bar{K}_2 - E^2 I^2 \lambda^2 K_1 K_2 - EI\lambda^3 (K_1 + K_2) \bar{K}_1 \bar{K}_2 - E^3 I^3 \lambda^7 (K_1 + \bar{K}_2)] A \\ + [EI\lambda (\bar{K}_1 + \bar{K}_2) (K_1 K_2 + E^2 I^2 \lambda^4) + EI\lambda^3 (K_1 + K_2) (\bar{K}_1 \bar{K}_2 + E^2 I^2 \lambda^2)] B \\ + [E^4 I^4 \lambda^8 + E^2 I^2 \lambda^4 (K_1 \bar{K}_1 + K_2 \bar{K}_2 + 2K_1 \bar{K}_2 + 2K_2 \bar{K}_1) + K_1 K_2 \bar{K}_1 \bar{K}_2] C \\ + 2E^2 I^2 \lambda^4 (\bar{K}_1 + \bar{K}_2) (K_1 + K_2) &= 0 \end{aligned} \quad (2.3)$$

其中:

$$\begin{aligned} A &= 2\sin \lambda L \cdot \operatorname{sh} \lambda L, \quad B = \sin \lambda L \cdot \operatorname{ch} \lambda L - \cos \lambda L \cdot \operatorname{sh} \lambda L, \\ C &= \cos \lambda L \cdot \operatorname{ch} \lambda L - 1 \end{aligned} \quad (2.4)$$

弹簧刚度系数 $K$ 在自由端和简支端的情况下分别取值为0和 $\infty$ ; 弹性固持座的刚度系数 $\bar{K}$ 在简支端和固持座的情况下分别取值为0和 $\infty$ 。如表所示。

表1  $K$ 及 $\bar{K}$ 值在不同支承条件下取值范围

刚度系数类别	支 承 条 件			
	自由	简支	固持	弹性支承
$K$	0	$\infty$	$\infty$	$0 < K < \infty$
$\bar{K}$	0	0	$\infty$	$0 < \bar{K} < \infty$

由(2.3)式可推得在各种边界条件下的频率方程。

1) 两端具有不同的弹簧支承(图1)

此时,  $\bar{K}_1 = \bar{K}_2 = 0$ 。(2.3)式可简化为:

$$K_1 K_2 A - EI\lambda^3 (K_1 + K_2) B - (EI\lambda^3)^2 C = 0 \quad (2.5)$$

2) 两端具有相同的弹簧支承(图5)。

此时,  $\bar{K}_1 = \bar{K}_2 = 0$ ,  $K_1 = K_2 = K$ 。(2.5)式可简化为:

$$K^2 A - 2EI\lambda^3 KB - (EI\lambda^3)^2 C = 0$$

可解得:

$$K = \frac{EI\lambda^3(B \pm \sqrt{B^2 + AC})}{A} \quad (2.6)$$

### 3) 两端简支 (图6)

此时,  $\bar{K}_1 = \bar{K}_2 = 0$ ,  $K_1 = K_2 = K = \infty$ 。由(2.6)式可得  $A = 0$ , 即

$$2\sin\lambda L \operatorname{sh}\lambda L = 0$$

上式中  $\operatorname{sh}\lambda L$  不为零, 故得频率方程为:

$$\sin\lambda L = 0 \quad (2.7)$$

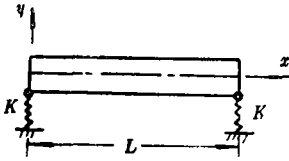


图 5

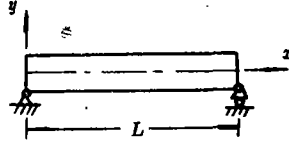


图 6

### 4) 两端自由 (图7)

此时  $\bar{K}_1 = \bar{K}_2 = 0$ ,  $K_1 = K_2 = K = 0$  由(2.5)式可得  $C = 0$ , 故得频率方程为:

$$\cos\lambda L \cdot \operatorname{ch}\lambda L = 1 \quad (2.8)$$

### 5) 一端简支, 一端自由 (图8)。

此时  $\bar{K}_1 = \bar{K}_2 = 0$ ,  $K_1 = \infty$ ,  $K_2 = 0$ 。将  $K_1$  除(2.5)式, 得

$$K_2 A - EI\lambda^3 \left(1 + \frac{K_2}{K_1}\right) B - (EI\lambda^3)^2 \frac{C}{K_1} = 0$$

把  $K_1$ 、 $K_2$  值代入上式, 得  $B = 0$ , 频率方程为:

$$\begin{aligned} \sin\lambda L \cdot \operatorname{ch}\lambda L - \cos\lambda L \cdot \operatorname{sh}\lambda L &= 0, \text{ 即} \\ \operatorname{tg}\lambda L &= \operatorname{th}\lambda L \end{aligned} \quad (2.9)$$

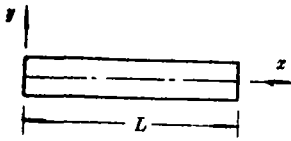


图 7

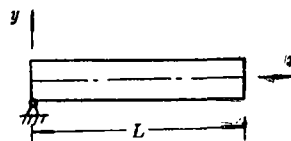


图 8

### 6) 一端弹簧支承, 一端简支 (图9)

此时  $\bar{K}_1 = \bar{K}_2 = 0$ ,  $K_1 = K$ ,  $K_2 = \infty$ 。将  $K_2$  除(2.5)式, 得:

$$K_1 A - EI\lambda^3 \left(\frac{K_1}{K_2} + 1\right) B - (EI\lambda^3)^2 \frac{C}{K_2} = 0$$

将  $K_2 = \infty$  代入上式, 得频率方程为:

$$K_1 = EI\lambda^3 \frac{B}{A} \quad (2.10)$$

7) 两端具有不同弹性固持座的梁(图2)。

此时  $K_1 = K_2 = \infty$ 。将  $K_1 K_2$  除以(2.3)式, 得频率方程为:

$$(\bar{K}_1 \bar{K}_2)C + EI\lambda(\bar{K}_1 + \bar{K}_2)B - (EI\lambda)^2 A = 0 \quad (2.11)$$

8) 两端具有相同弹性固持座(图10)

此时,  $K_1 = K_2 = \infty$ ,  $\bar{K}_1 = \bar{K}_2 = \bar{K}$ 。(2.11)式可改写为:

$$\bar{K}^2 C + 2EI\lambda \bar{K} B - (EI\lambda)^2 A = 0$$

可解得:

$$\bar{K} = EI\lambda \left( \frac{-B \pm \sqrt{B^2 + AC}}{C} \right) \quad (2.12)$$

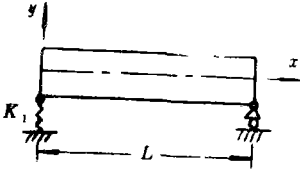


图 9

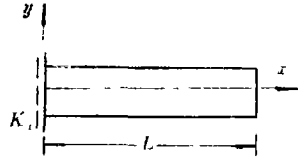


图 10

9) 两端固持(图11)

此时,  $K_1 = K_2 = \infty$ ,  $\bar{K}_1 = \bar{K}_2 = \bar{K} = \infty$ , 由(2.12)式可得  $C = 0$ , 即

$$\cos \lambda L \cdot \operatorname{ch} \lambda L = 1 \quad (2.13)$$

与两端自由梁的方程(2.8)相同。

10) 一端固持, 一端简支(图12)。

此时,  $K_1 = K_2 = \infty$ ,  $\bar{K}_1 = \infty$ ,  $\bar{K}_2 = 0$ 。将  $\bar{K}_1$  除(2.11)式, 得

$$\bar{K}_2 C + EI\lambda \left( 1 + \frac{\bar{K}_2}{\bar{K}_1} \right) B - (EI\lambda)^2 \frac{A}{\bar{K}_1} = 0$$

$\therefore B = 0$ , 即

$$\operatorname{tg} \lambda L = \operatorname{th} \lambda L \quad (2.14)$$

与一端简支, 一端自由梁的方程(2.9)相同。

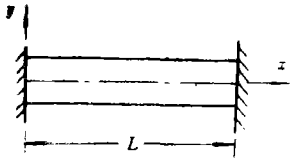


图 11

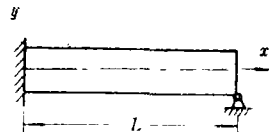


图 12

11) 一端弹性固持, 一端简支(图13)。

此时,  $K_1 = K_2 = \infty$ ,  $\bar{K}_1 = \bar{K}$ ,  $\bar{K}_2 = 0$ 。由(2.11)式得:

$$\bar{K} B - EI\lambda A = 0$$

$$\therefore \bar{K} = EI\lambda \frac{A}{B} \quad (2.15)$$

12) 一端弹性固持, 一端固持 (图14)。

此时,  $K_1 = K_2 = \infty, \bar{K}_1 = \bar{K}, \bar{K}_2 = \infty$ . 将  $\bar{K}_2$  除以(2.11)式, 得:

$$\bar{K}_1 C + EI\lambda \left( \frac{\bar{K}_1}{\bar{K}_2} + 1 \right) B - (EI\lambda)^2 \frac{A}{\bar{K}_2} = 0$$

$$\therefore \bar{K} = -EI\lambda \frac{B}{C} \quad (2.16)$$

13) 一端弹性固持, 一端弹簧支承 (图3)。

此时,  $K_1 = \infty, \bar{K}_1 = \bar{K}, K_2 = K, \bar{K}_2 = 0$ . 用  $K_1$  除以(2.3)式, 并将  $K_1, \bar{K}_1, K_2, \bar{K}_2$  各值代入, 化简后得:

$$-EI\lambda K_2 A + [\bar{K}_1 K_2 + (EI\lambda^2)^2] B + \bar{K}_1 EI\lambda^3 (C + 2) = 0 \quad (2.17)$$

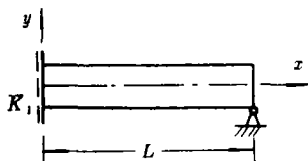


图 13

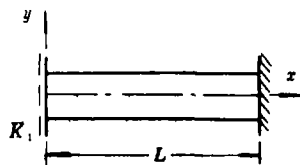


图 14

14) 悬臂梁 (图15),

此时,  $K_1 = \infty, \bar{K}_1 = \infty, K_2 = 0, \bar{K}_2 = 0$ . 用  $\bar{K}_1$  除(2.17)式, 得:

$$-EI\lambda \frac{K_2}{\bar{K}_1} A + \left[ K_2 + \frac{(EI\lambda^2)^2}{\bar{K}_1} \right] B + EI\lambda^3 (C + 2) = 0$$

$$\therefore C + 2 = 0, \text{ 即}$$

$$\cos \lambda L \cdot \operatorname{ch} \lambda L + 1 = 0 \quad (2.18)$$

15) 一端弹性固持, 一端自由 (图16)

此时,  $K_1 = \infty, \bar{K}_1 = \bar{K}, K_2 = 0, \bar{K}_2 = 0$  代入(2.17)式, 得:

$$EI\lambda B + \bar{K}_1 (C + 2) = 0$$

$$\therefore \bar{K} = -EI\lambda \frac{B}{(C + 2)} \quad (2.19)$$

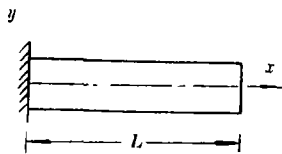


图 15

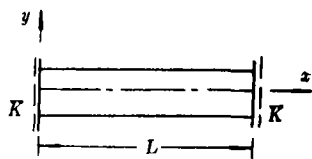


图 16

16) 一端固持, 一端弹簧支承 (图17)

此时,  $K_1 = \bar{K}_1 = \infty, K_2 = K, \bar{K}_2 = 0$ , 用  $\bar{K}_1$  除以(2.17)式, 得:

$$K_2 B + EI\lambda^3 (C + 2) = 0$$

$$\therefore K = -EI\lambda^3 \frac{(C + 2)}{B} \quad (2.20)$$

17) 一端弹簧支承, 一端自由 (图18)

此时,  $\bar{K}_1 = \bar{K}_2 = 0, K_1 = K, K_2 = 0$ 。代入(2.5)式, 得:

$$-EI\lambda^3 KB - (EI\lambda^3)^2 C = 0$$

$$\therefore K = -EI\lambda^3 \frac{C}{B} \quad (2.21)$$

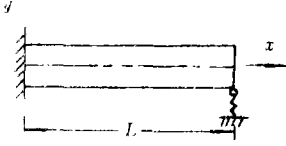


图 17

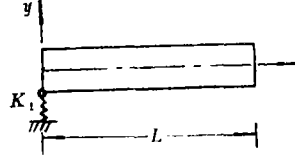


图 18

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## Connecting Body Calculation Operated By The Hand of The Robot

Gu Langping

### Abstract

An important theoretical problem of the robot is how to gain necessary movement by controlling mechanical hand. This paper describes the basic movement of the manual operation as the system of a rigid body which is made up of seven parts with nine degrees of freedom. These movements are analysed strictly and exactly with the gravity, control and friction of the seven parts into consideration. Various functions can be cleverly performed by the robot. In order to facilitate calculation on the computer, movement differential equations are created in terms of Kane equation. These differential equations are expressed by generalized coordinates, which can be regarded as theoretical proof of analysing the movement mechanical hand.

**Key words** Partial velocity, Partial angular velocity, general velocity, general active force, General inertial force

## The General Expression for the lateral Vibration Frequency Equations of Beams

Zhuo Shujun Ge Yujun

### Abstract

The general expression for the flexural vibration frequency equations of beams is derived in this paper, Seventeen lateral frequency vibration equations under various supporting conditions are all its special cases.

**Key words** Beams, Vibration, Structural vibration, Equation of frequency