

任意三角域上的光滑插值

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摘 要 文中描述了任意三角域上插值逼近的两种方法,一种是布尔和插值,一种是凸组合插值,其特点是结构简单,易于计算。文中证明了三角网格区域上的整个插值曲面是 C^1 连续的,并指出离散化的插值曲面函数是空间散乱数据的一种光滑插值算法。

关键词 几何, 计算, 插值; 布尔和, 凸组合, 离散化

分类号 O174.42

引 言

三角曲面片是曲面造型的基本单元之一,特别是一些有拐角的曲面,三角片必不可少,如汽车车身外形设计许多地方就要用到,这说明研究三角曲面片有很大的实用价值。

三角形上插值曲面的研究,基本上可归为三类:(1)先在标准三角形上的插值^[1],然后将其仿射变换成任意三角形上的插值,这种插值曲面计算繁杂^[2];(2)以重心坐标研究三角形上的插值曲面^[3],但其实质与(1)相同,只是表现形式不同而已,不过从自由造型的角度来看,这种方法仍是较好的^[4];(3)在任意三角形上建立局部坐标系,在局部坐标系下研究插值逼近^{[5][6]},这种方法结构简便,易于计算。

本文描述局部坐标系下两种不同于^{[5][6]}的插值逼近方法,给出了误差估计和代数精度集,其中布尔和插值误差比^[5]小,代数精度集包含了^[5]中的代数精度集。在同一坐标系下,证明了相邻三角形上的曲面片是 C^1 拼接的,从而整个的曲面可以在同一坐标系下方便地表示。

1 三角域上两种插值曲面的构造

1.1 布尔和插值逼近

设 $\triangle ABC$ 的较长边为 AB ,取 AB 作为 y 轴,建立局部坐标系如图(1)。

记

$$x_0 = |OC|, y_1 = |OA|, y_2 = -|OB|,$$

$\triangle ABC$ 为 T ,它的边界为 ∂T 。

如果三角形 T 上的函数 $F(x, y)$ 给定,我们希望找到一

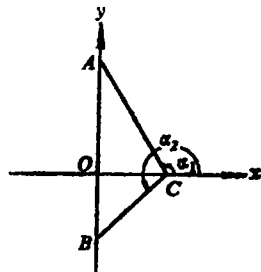


图 1

个光滑的有理函数 $G(x, y)$, 使 $G(x, y)$ 在 ∂T 上插值 F 及 F 的一阶偏导数, 即

$$\left. \begin{aligned} G(x, y)|_{\partial T} &= F(x, y)|_{\partial T} \\ \frac{\partial G(x, y)}{\partial x} \Big|_{\partial T} &= \frac{\partial F(x, y)}{\partial x} \Big|_{\partial T}, \quad \frac{\partial G(x, y)}{\partial y} \Big|_{\partial T} = \frac{\partial F(x, y)}{\partial y} \Big|_{\partial T} \end{aligned} \right\} \quad (1)$$

设 $\phi_i(t), \varphi_i(t)$ 是区间 $[0, 1]$ 上的两点 Hermite 插值基函数

$$\phi_0(t) = (t-1)^2(2t-1), \quad \phi_1(t) = t(t-1)^2$$

$$\varphi_0(t) = t^2(3-2t), \quad \varphi_1(t) = (t-1)t^2$$

记

$$F_{i,j}(x, y) = \frac{\partial F(x, y)}{\partial x^i \partial y^j}$$

文[5]中构造了平行 OC 的线性算子

$$P_1 F(x, y) = \sum_{i=0}^1 \phi_i(x/m) m^i F_{i,0}(0, y) + \sum_{i=0}^1 \varphi_i(x/m) m^i F_{i,0}(m, y) \quad (2)$$

其中

$$\begin{aligned} m &= m(y) \\ &= x_0 - y \operatorname{ctg} \alpha_j, \quad j = \begin{cases} 1 & y \geq 0 \\ 2 & y < 0 \end{cases} \end{aligned}$$

显然函数 $P_1 F(x, y)$ 在 $\triangle ABC$ 的三边上插值 F 及 F 的一阶偏导数, 但在 OC 上关于 y 的偏导数不连续, 从而 $P_1 F$ 在 T 上的光滑性得不到保证。文[5]添加一有理多项式消除了这种不连续现象。我们用布尔和与凸组合插值方法不仅能在 ∂T 上插值 F 及 F 的一阶偏导数, 而且在 OC 上也能达到同样的要求。

我们先引入 OC 上的泰勒算子

$$P_2 F(x, y) = F(x, 0) + y F_{0,1}(x, 0) \quad (3)$$

令

$$G(x, y) = (P_1 \oplus P_2) F(x, y) = P_1 F(x, y) + P_2 F(x, y) - P_1 P_2 F(x, y) \quad (4)$$

下面来证明 $G(x, y)$ 就是我们所要寻找的函数。

由[5]知

$$\begin{aligned} P_1 F|_{\partial T} &= F|_{\partial T} \\ \frac{\partial P_1 F}{\partial x} \Big|_{\partial T} &= \frac{\partial F}{\partial x} \Big|_{\partial T}, \quad \frac{\partial P_1 F}{\partial y} \Big|_{\partial T} = \frac{\partial F}{\partial y} \Big|_{\partial T} \end{aligned}$$

由(4)式和以上条件可得

$$\begin{aligned} G(x, y)|_{\partial T} &= F(x, y)|_{\partial T} + P_2 F(x, y)|_{\partial T} - P_2 F(x, y)|_{\partial T} \\ &= F(x, y)|_{\partial T} \end{aligned}$$

同理

$$\frac{\partial G(x, y)}{\partial x} \Big|_{\partial T} = \frac{\partial F(x, y)}{\partial x} \Big|_{\partial T}, \quad \frac{\partial G(x, y)}{\partial y} \Big|_{\partial T} = \frac{\partial F(x, y)}{\partial y} \Big|_{\partial T}$$

设 I 为单位算子 (即 $IF = F$), 则

$$G(x, y) = P_2 F(x, y) + P_1(I - P_2)F(x, y)$$

$$\begin{aligned}
 &= P_2 F(x, y) + \sum_{i=0}^1 \phi_i(x/m) m^i [F_{i_0}(0, y) - F_{i_0}(0, 0) - y F_{i_1}(0, 0)] \\
 &+ \sum_{i=0}^1 \varphi_i(x/m) m^i [F_{i_0}(m, y) - F_{i_0}(m, 0) - y F_{i_1}(m, 0)] \quad (5)
 \end{aligned}$$

由上式可得

$$\begin{aligned}
 G(x, y)|_{OC} &= G(x, y)|_{y=0} \\
 &= F(x, 0) \\
 \left. \frac{\partial G(x, y)}{\partial y} \right|_{OC} &= \left. \frac{\partial G(x, y)}{\partial y} \right|_{y=0} \\
 &= F_{01}(x, 0) \quad (6)
 \end{aligned}$$

于是我们得到如下定理。

定理1 设 $F_{11} \in C^2$, 则布尔和插值逼近函数 $G(x, y)$ 在 ∂T 及 OC 上插值 F 及 F 的一阶偏导数。

设 $G(x, y)$ 插值逼近 $F(x, y)$ 的误差为 $E(x, y)$, 则

$$\begin{aligned}
 E(x, y) &= |G(x, y) - F(x, y)| \\
 &= |P_1 F(x, y) + P_2 F(x, y) - P_1 P_2 F(x, y) - F(x, y)| \\
 &= |(I - P_1)(I - P_2)F(x, y)|
 \end{aligned}$$

由泰勒展开的余项公式, 从(3)式得

$$\begin{aligned}
 (I - P_2)F(x, y) &= F(x, y) - P_2 F(x, y) \\
 &= \int_0^y \frac{\partial^2 F(x, t)}{\partial t^2} (y - t) dt
 \end{aligned}$$

当 y 固定时, $G(x, y)$ 是关于 x 的代数多项式, 且在点 $(0, y)$, $(m(y), y)$ 处插值 F 及 F 的一阶偏导数, 于是根据 Hermite 插值误差估计得

$$\begin{aligned}
 E(x, y) &= \left| \frac{x^2(x-m)^2}{4!} \int_0^y \frac{\partial^6 F(\theta x, t)}{\partial x^4 \partial t^2} (y-t) dt \right| \\
 &\leq \frac{x^2 y^2 (x-m)^2}{48} M_6 \quad (7)
 \end{aligned}$$

其中

$$M_6 = \text{Max}_{(x, y) \in T} \left\{ \frac{\partial^6 F(x, y)}{\partial x^4 \partial y^2} \right\} \quad 0 < \theta < x \quad (8)$$

从(7)式可直接得到代数精度集。

推论1 布尔和插值逼近的代数精度集

$$\tau = \{x^i y^j; i > 0, j > 0, i + j \leq 5\}$$

1.2 凸组合插值逼近

我们仔细检查定理1的证明过程会发现 $G(x, y)$ 必须满足称为相容性条件的下式

$$\frac{\partial^2 F(0, 0)}{\partial x \partial y} = \frac{\partial^2 F(0, 0)}{\partial y \partial x}, \quad \frac{\partial^2 F(x_0, 0)}{\partial x \partial y} = \frac{\partial^2 F(x_0, 0)}{\partial y \partial x} \quad (9)$$

条件(9)式是由布尔和插值中 $P_1 P_2 F$ 所产生, 要消除这些条件, 我们引入一种凸组

合插值方法。

给出两个非负函数

$$\alpha_1(x, y) = \frac{y^2}{y^2 + x^2(x-m)^2}, \quad \alpha_2(x, y) = \frac{x^2(x-m)^2}{y^2 + x^2(x-m)^2} \quad (10)$$

显然 $\alpha_1(x, y) + \alpha_2(x, y) = 1$

我们称函数

$$G(x, y) = \alpha_1(x, y)P_1F(x, y) + \alpha_2(x, y)P_2F(x, y) \quad (11)$$

为 $F(x, y)$ 的凸组合插值逼近。

由 $\alpha_1(x, y)$, $\alpha_2(x, y)$ 易得

$$\begin{aligned} G(x, y)|_{\partial T} &= F(x, y)|_{\partial T} \\ G(x, y)|_{OC} &= G(x, y)|_{y=0} \\ &= F(x, 0) \end{aligned}$$

注意到 $\alpha_1(x, y)$, $\alpha_2(x, y)$ 对 x, y 的偏导数, 可得

$$\begin{aligned} \left. \frac{\partial G(x, y)}{\partial x} \right|_{\partial T} &= \left. \frac{\partial \alpha_1(x, y)P_1F(x, y)}{\partial x} \right|_{\partial T} \\ &= \left. \frac{\partial P_1F(x, y)}{\partial x} \right|_{\partial T} \\ &= \left. \frac{\partial F(x, y)}{\partial x} \right|_{\partial T} \end{aligned}$$

同理

$$\left. \frac{\partial G(x, y)}{\partial y} \right|_{\partial T} = \left. \frac{\partial F(x, y)}{\partial y} \right|_{\partial T}$$

而

$$\begin{aligned} \left. \frac{\partial G(x, y)}{\partial y} \right|_{OC} &= \left. \frac{\partial \alpha_2(x, y)P_2F(x, y)}{\partial y} \right|_{y=0} \\ &= \left. \frac{\partial P_2F(x, y)}{\partial y} \right|_{y=0} \\ &= F_{01}(x, 0) \end{aligned}$$

因此我们有类似定理1的如下定理。

定理2 设 $F_{11} \in C^2$, 非负函数 $\alpha_1(x, y)$, $\alpha_2(x, y)$ 由(10)式给出, 则凸组合插值逼近函数 $G(x, y)$ 在 ∂T 及 OC 上插值 F 及 F 的一阶偏导数。

设凸组合插值函数的逼近误差为 $E(x, y)$, 则

$$\begin{aligned} E(x, y) &= |G(x, y) - F(x, y)| \\ &= |\alpha_1(x, y)(I - P_1)F(x, y) + \alpha_2(x, y)(I - P_2)F(x, y)| \end{aligned}$$

用类似布尔和误差估计的方法得

$$E(x, y) \leq \left| \frac{y^2}{y^2 + x^2(x-m)^2} \frac{x^2(x-m)^2}{4!} F_{40}(\theta, x, y) \right|$$

$$\begin{aligned}
 & + \frac{x^2(x-m)^2}{y^2+x^2(x-m)^2} y^2 \frac{F_{02}(x, \theta_2, y)}{2!} \Big| \\
 & \leq x^2(x-m)^2 (M_4/4! + M_2/2!) \tag{12}
 \end{aligned}$$

其中

$$0 < \theta_1 < x_1, \quad y_2 < \theta_2 < y_1$$

$$M_2 = \text{Max}_{(x,y) \in T} \left\{ \frac{\partial^2 F(x,y)}{\partial y^2} \right\}, \quad M_4 = \text{Max}_{(x,y) \in T} \left\{ \frac{\partial^4 F(x,y)}{\partial x^4} \right\}$$

推论 2 凸组合插值逼近函数的代数精度集

$$\tau = \{x^i y^j : 0 \leq i \leq 3, 0 \leq j \leq 1\}$$

2 相邻两三角域上的曲面片 C^1 拼接的证明

我们选择一个固定的坐标系 $o-xyz$, 设 D 是 xy 平面上的一个闭区域, $F(x, y)$ 是 D 上一个二元函数, $P_i = \{x_i, y_i\}_{i=1}^n$ 是 D 中的散乱点, 将这些点三角网格化, 则在每个三角形上按上节的方法可求出插值曲面函数, 显然相邻两曲面片是 C^0 连续的, 而我们的目的是要证明它们是 C^1 连续的。

设 $P_L = \{x_L, y_L\} (L=i, j, k)$ 是右手系 $\Delta P_i P_j P_k$ 的三个顶点, 记 $\Delta P_i P_j P_k$ 为 T_1 , 它的顶点对应边 E_L 的外法向量为 $n_L (L=i, j, k)$, T_1 上的局部坐标系为 $\bar{o}-\bar{x}\bar{y}\bar{z}$; 又设 P_o 是 T_1 邻近一点, $\Delta P_j P_o P_k$ 是一个与 T_1 相邻的右手系三角形, 记为 T_2 , 它与 T_1 的公共边为 E_i , 它对应的局部坐标系是 $o'-x'y'z'$, 如图 2 所示。

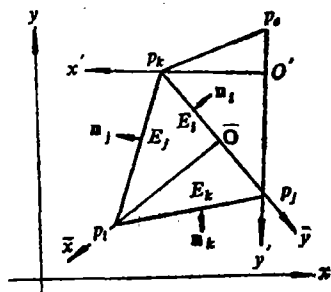


图 2

记 $F(x, y)$ 在 $\bar{o}-\bar{x}\bar{y}\bar{z}$ 和 $o'-x'y'z'$ 中的函数表示分别为 $\bar{F}(\bar{x}, \bar{y})$, $F^*(x', y')$, 而在 T_1, T_2 上的插值逼近函数分别为 $\bar{G}(\bar{x}, \bar{y})$, $G^*(x', y')$, 且设

$$G1(x, y) = \bar{G}(\bar{x}, \bar{y}), \quad G2(x, y) = G^*(x', y')$$

再由坐标变换公式知坐标系之间有如下关系

$$\begin{cases} x - \bar{x}_0 = \bar{x} \cos \theta_1 - \bar{y} \sin \theta_1 \\ y - \bar{y}_0 = \bar{x} \sin \theta_1 + \bar{y} \cos \theta_1 \end{cases}$$

$$\begin{cases} x - x'_0 = x' \cos \theta_2 - y' \sin \theta_2 \\ y - y'_0 = x' \sin \theta_2 + y' \cos \theta_2 \end{cases}$$

其中 θ_1, θ_2 分别是 \bar{x} 轴和 x' 轴与 x 轴的有向夹角, $(\bar{x}_0, \bar{y}_0), (x'_0, y'_0)$ 分别是 \bar{o}, o' 在 $o-xy$ 平面上的坐标。

由上节的插值表明满足如下条件

$$\left. \begin{aligned} \frac{\partial \bar{G}}{\partial \bar{x}} \Big|_{E_i} &= \frac{\partial \bar{F}}{\partial \bar{x}} \Big|_{E_i}, \quad \frac{\partial \bar{G}}{\partial \bar{y}} \Big|_{E_i} = \frac{\partial \bar{F}}{\partial \bar{y}} \Big|_{E_i} \\ \frac{\partial G^*}{\partial x'} \Big|_{E_i} &= \frac{\partial F^*}{\partial x'} \Big|_{E_i}, \quad \frac{\partial G^*}{\partial y'} \Big|_{E_i} = \frac{\partial F^*}{\partial y'} \Big|_{E_i} \end{aligned} \right\} \tag{12}$$

由法向导数公式知

$$\frac{\partial G}{\partial n_i} = \frac{y_k - y_j}{\sqrt{d_i}} \frac{\partial G}{\partial x} + \frac{x_j - x_k}{\sqrt{d_i}} \frac{\partial G}{\partial y}$$

其中 d_i 表示 E_i 的长度, 所以有

$$\left. \begin{aligned} \frac{\partial G_1(x, y)}{\partial n_i} &= \frac{y_k - y_j}{\sqrt{d_i}} \frac{\partial G_1(x, y)}{\partial x} + \frac{x_j - x_k}{\sqrt{d_i}} \frac{\partial G_1(x, y)}{\partial y} \\ \frac{\partial G_2(x, y)}{\partial n_i} &= \frac{y_k - y_j}{\sqrt{d_i}} \frac{\partial G_2(x, y)}{\partial x} + \frac{x_j - x_k}{\sqrt{d_i}} \frac{\partial G_2(x, y)}{\partial y} \end{aligned} \right\} \quad (13)$$

而

$$\begin{aligned} \frac{\partial G_1(x, y)}{\partial x} &= \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial x} \\ &= \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial x} \\ \frac{\partial G_1(x, y)}{\partial y} &= \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial y} \\ &= \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial y} + \frac{\partial \bar{G}(\bar{x}, \bar{y})}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y} \end{aligned}$$

将(12)式的第一式代入上式得

$$\left. \begin{aligned} \frac{\partial G_1(x, y)}{\partial x} \Big|_{E_i} &= \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{x}} \Big|_{E_i} \frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{y}} \Big|_{E_i} \frac{\partial \bar{y}}{\partial x} \\ \frac{\partial G_1(x, y)}{\partial y} \Big|_{E_i} &= \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{x}} \Big|_{E_i} \frac{\partial \bar{x}}{\partial y} + \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{y}} \Big|_{E_i} \frac{\partial \bar{y}}{\partial y} \end{aligned} \right\} \quad (14)$$

再将

$$\begin{aligned} \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{x}} &= \frac{\partial F(x, y)}{\partial x} \\ &= \frac{\partial F(x, y)}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial F(x, y)}{\partial y} \frac{\partial y}{\partial \bar{x}} \\ \frac{\partial \bar{F}(\bar{x}, \bar{y})}{\partial \bar{y}} &= \frac{\partial F(x, y)}{\partial y} \\ &= \frac{\partial F(x, y)}{\partial x} \frac{\partial x}{\partial \bar{y}} + \frac{\partial F(x, y)}{\partial y} \frac{\partial y}{\partial \bar{y}} \end{aligned}$$

代入(14)式得:

$$\begin{aligned} \frac{\partial G_1(x, y)}{\partial x} \Big|_{E_i} &= \left(\frac{\partial x}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} + \frac{\partial x}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial x} \right) \frac{\partial F(x, y)}{\partial x} \Big|_{E_i} \\ &\quad + \left(\frac{\partial y}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} + \frac{\partial y}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial x} \right) \frac{\partial F(x, y)}{\partial y} \Big|_{E_i} \end{aligned}$$

$$\begin{aligned}
 &= (\cos^2 \theta_1 + \sin^2 \theta_1) \frac{\partial F(x, y)}{\partial x} \Big|_{E_i} \\
 &+ (\sin \theta_1 \cos \theta_1 - \cos \theta_1 \sin \theta_1) \frac{\partial F(x, y)}{\partial y} \Big|_{E_i} \\
 &= \frac{\partial F(x, y)}{\partial x} \Big|_{E_i}
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 \frac{\partial G_1(x, y)}{\partial y} \Big|_{E_i} &= (-\sin \theta_1 \cos \theta_1 + \sin \theta_1 \cos \theta_1) \frac{\partial F(x, y)}{\partial x} \Big|_{E_i} \\
 &+ (\sin^2 \theta_1 + \cos^2 \theta_1) \frac{\partial F(x, y)}{\partial y} \Big|_{E_i} \\
 &= \frac{\partial F(x, y)}{\partial y} \Big|_{E_i}
 \end{aligned} \tag{16}$$

同理可证

$$\frac{\partial G_2(x, y)}{\partial x} \Big|_{E_i} = \frac{\partial F(x, y)}{\partial x} \Big|_{E_i}, \quad \frac{\partial G_2(x, y)}{\partial y} \Big|_{E_i} = \frac{\partial F(x, y)}{\partial y} \Big|_{E_i} \tag{17}$$

将(15), (17)式代入(13)式得

$$\begin{aligned}
 \frac{\partial G_1(x, y)}{\partial n_i} \Big|_{E_i} &= \frac{\partial G_2(x, y)}{\partial n_i} \Big|_{E_i} \\
 &= \frac{\partial F(x, y)}{\partial n_i} \Big|_{E_i}
 \end{aligned}$$

同样的道理可知：以 E_j, E_k 为公共边的相邻三角曲面片也满足(17)式，因而我们证明了每相邻两三角形上的曲面片是 C^1 连续的，故三角网格上的整个曲面也是 C^1 连续的。

对于空间散乱数据的插值，只要将前面的插值函数离散化^[6]，就是一个 C^1 插值算法。

3 数值例子

用本文所研究的两个插值逼近算法，我们在 $\triangle ABC$ 上绘制了一组原函数图和逼近函数图，如图 3 所示，表明比较逼近的程度。

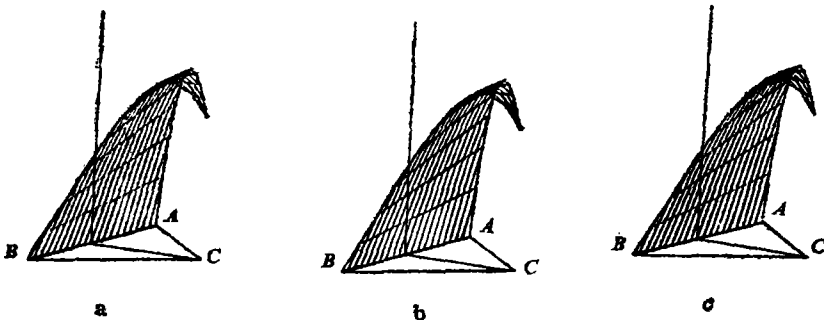


图 3

a 函数图

b 布尔和插值逼近图

c 凸组合插值逼近图

例 设 $F(x, y) = \sin(2\pi x(1 - y))$

$$A = (0, 0.5), B = (0, -0.5), C = (0.4, 0)$$

从图 3 上看, 似乎没有什么差别, 这正好说明了我们的插值函数很好地逼近了原函数, 而与理论值的误差是相符合的。

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Smooth Interpolation in Arbitrary Triangles

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Abstract

In this paper two interpolating approximation schemes in arbitrary triangles are described; one is Boolean sum, the other is convex combination. These interpolated surfaces are simple to construct and easy to calculate. It is shown that the whole interpolated surface in triangular domain is C^1 continuous. Finally, it is pointed out that the discretization interpolating surface is a C^1 interpolating surface for scattered 3D data.

Key Words: geomtry, computing, interpolation; Boolean sum, convex combination, discretization