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一类集值映射的高阶微分

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摘　要　在文[1]中，H.T.Banks与M.Q.Jacobs讨论了集映$\Omega:R^m\to\mathscr{B}(R^n)$的一阶微分，分别给出了$\Omega$可微的充分条件与必要条件。本文改进了文[1]的工作，证明了[1]中条件(3.2)的限制是不必要的，从而得到了Ω可微的充分必要条件，及较为简洁的Ω微分解析表达式。在文中也讨论了Ω的k阶微分，得到了Ωk阶可微的充分必要条件及k阶微分的解析表达式。

关键词　集值映射，Hausdorff距离，集映微分，集映高阶微分

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1　预备知识

R^n上的范数取作$\|x\|=\max\limits_{1<j<n}|x_j|$，$R^n$中非空紧凸集全体按Hausdorff距离$d_H$所成的度量空间记作$\mathscr{B}(R^n)$，($d_H(A,B)\triangleq\max\{\sup\limits_{x\in A}d(x,B),\ \sup\limits_{y\in B}d(y,A)\}$)，根据Radstrom嵌入

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An Unconstrained Optimization Technique —Proper Conjugate Direction Method

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Abstract

In this paper, a method for solving unconstrained optimization problems, which is called "proper conjugate direction method", is developed. Compared with general conjugate direction method,such as the conjugate gradient method, this method will save a great deal of one-dimensional search work and raise the rate of convergence.

Key words: convergence, unconstrained optimizatien problem, proper conjugate direction method, conjugate gradient method, one-dimensional search

定理([2])，$\mathscr{B}(R^n)$可以被等距同构地嵌入到一个线性赋范空间$\boldsymbol{B}(R^n)$中去，为后面叙述方便，在此对嵌入过程作一简单介绍。

在$\mathscr{B}(R^n)\times\mathscr{B}(R^n)$上定义一个等价关系如下：

$$(x_1,y_1)\sim(x_2,y_2)\Longleftrightarrow x_1+y_2=x_2+y_1$$

令$\boldsymbol{B}(R^n)=\mathscr{B}(R^n)\times\mathscr{B}(R^n)/\sim$，其中元素记作$\langle x,y\rangle$，$(x,y\in\mathscr{B}(R^n))$，$\boldsymbol{B}(R^n)$中范数定义为：

$$\|\langle x,y\rangle\|=d_H(x,y)$$

加法与数乘定义为：

$$\langle x_1,y_1\rangle+\langle x_2,y_2\rangle=\langle x_1+x_2,y_1+y_2\rangle$$

$$a\langle x,y\rangle=\begin{cases}\langle ax, & ay\rangle,\ a\geqslant 0\text{时}\\ \langle -ay, & -ax\rangle,\ a<0\text{时}\end{cases}$$

于是$\boldsymbol{B}(R^n)$成为一线性赋范空间。又定义$\pi:\mathscr{B}(R^n)\to\boldsymbol{B}(R^n)$，$\pi(x)=\langle x,0\rangle$，$x\in\mathscr{B}(R^n)$. 容易验证$\pi$是一个等距线性嵌入。$\forall A\in\mathscr{B}(R^n)$，$\pi(A)\triangleq\hat{A}$.

对于通常的m元实值函数$u:R^m\to R^1$，约定如下微分记号：

$$Du(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i\frac{\partial u}{\partial x^i}(x_0)\triangleq\left(\sum_{i=1}^{m}\Delta x^i\frac{\partial}{\partial x^i}\right)u(x_0)\quad(\Delta x\in R^m)$$

$$D^2u(x_0)(\Delta x)\triangleq\left(\sum_{i=1}^{m}\Delta x^i\frac{\partial}{\partial x^i}\right)^2u(x_0)$$

一般地

$$D^ru(x_0)(\Delta x)\triangleq\left(\sum_{i=1}^{m}\Delta x^i\frac{\partial}{\partial x^i}\right)^ru(x_0)$$

给定一个集映$\Omega:R^m\to\mathscr{B}(R^n)$，称$\Omega$在$x_0\in R^m$处$\pi$-可微是指：$\hat{\Omega}\triangleq\pi(\Omega):R^m\to\boldsymbol{B}(R^n)$是可微的，即存在一个连续的线性映射$D\hat{\Omega}(x_0):R^m\to\boldsymbol{B}(R^n)$使得

$$\hat{\Omega}(x)-\hat{\Omega}(x_0)-D\hat{\Omega}(x_0)(x-x_0)=o(\|x-x_0\|)\quad(x\to x_0)$$

称$D\hat{\Omega}(x_0)(\Delta x)$为$\Omega$在$x_0$处（关于自变量增量$\Delta x$）的一阶微分。

最后引入两个记号如下：

$$\mathscr{B}_1(R^n)\triangleq\left\{A\in\mathscr{B}(R^n)\,\middle|\,A=\prod_{j=1}^{n}[a_j,b_j],\ a_j,b_j\in R^1\right\}$$

$$\boldsymbol{B}_1(R^n)\triangleq\{\langle A,B\rangle\in\boldsymbol{B}(R^n)\,|\,A,B\in\mathscr{B}_1(R^n)\}$$

2　主要结果

关于集映$\Omega:R^m\to\mathscr{B}_1(R^n)$，$\Omega(x)=\prod_{j=1}^{n}[a^j(x),b^j(x)]$，其一阶微分由[1]中定理3.1，定理3.2已有如下结果：

若Ω在$x_0\in R^m$处π-可微，由$D\hat{\Omega}(x_0)$线性知，$\exists A_i(x_0)$，$B_i(x_0)\in\mathscr{B}(R^n)$使得

$$D\hat{\Omega}(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i\langle A_i(x_0),B_i(x_0)\rangle\quad(\Delta x\in R^m)\tag{1}$$

若进一步假设有

$$\langle A_i(x_0),\ B_i(x_0)\rangle\in\boldsymbol{B}_1(R^n)\quad(i=1,2,\cdots,n)\tag{2}$$

则a^j，$b^j: R^m \to R^1$在x_0处是可微的，$j=1,2,\cdots,n$.

反之，若a^j，$b^j: R^m \to R^1$在x_0处是可微的，则Ω在x_0处是π-可微的，且

$$D\hat{\Omega}(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i\left\langle \prod_{j=1}^{n}\left[\alpha_i^j(x_0)+\frac{\partial a^j}{\partial x^i}(x_0),\ \beta_i^j(x_0)+\frac{\partial b^j}{\partial x^i}(x_0)\right],\right.$$
$$\left.\prod_{j=1}^{n}[\alpha_i^j(x_0),\ \beta_i^j(x_0)]\right\rangle \tag{3}$$

式中$\alpha_i^j(x_0)$，$\beta_i^j(x_0)$是满足下述不等式的任意实数：

$$\alpha_i^j(x_0)\leqslant\beta_i^j(x_0)\qquad i=1,2,\cdots,m;\ j=1,2,\cdots,n \tag{4}$$

$$\alpha_i^j(x_0)+\frac{\partial a^j}{\partial x^i}(x_0)\leqslant\beta_i^j(x_0)+\frac{\partial b^j}{\partial x^i}(x_0)\quad i=1,2,\cdots,m;\ j=1,2,\cdots,n \tag{5}$$

先将(3)式右边写成较为简洁的形式，接着证明条件(2)式是不必要的。

记

$$a_i^j\triangleq\frac{\partial a^j}{\partial x^i}(x_0),\ b_i^j\triangleq\frac{\partial b^j}{\partial x^i}(x_0)$$

$$\Delta x_+^i=\begin{cases}\Delta x^i, & \Delta x^i>0\text{时}\\ 0, & \Delta x^i\leqslant 0\text{时}\end{cases}\qquad \Delta x_-^i=\begin{cases}0, & \Delta x^i>0\text{时}\\ -\Delta x^i, & \Delta x^i\leqslant 0\text{时}\end{cases}$$

于是

$$\begin{aligned}&\sum_{i=1}^{m}\Delta x^i\left\langle\prod_{j=1}^{n}[\alpha_i^j+a_i^j,\ \beta_i^j+b_i^j],\ \prod_{j=1}^{n}[\alpha_i^j,\ \beta_i^j]\right\rangle\\
&=\sum_{i=1}^{m}\Delta x_+^i\left\langle\prod_{j=1}^{n}[\alpha_i^j+a_i^j,\ \beta_i^j+b_i^j],\ \prod_{j=1}^{n}[\alpha_i^j,\ \beta_i^j]\right\rangle+\\
&+\sum_{i=1}^{m}\Delta x_-^i\left\langle\prod_{j=1}^{n}[\alpha_i^j,\ \beta_i^j],\ \prod_{j=1}^{n}[\alpha_i^j+a_i^j,\ \beta_i^j+b_i^j]\right\rangle\\
&=\left\langle\prod_{j=1}^{n}\left[\sum_{i=1}^{m}(\Delta x^i\alpha_i^j+\Delta x_+^i a_i^j),\ \sum_{i=1}^{m}(\Delta x^i\beta_i^j+\Delta x_+^i b_i^j)\right],\right.\\
&\qquad\left.\prod_{j=1}^{n}\left[\sum_{i=1}^{m}(\Delta x^i\alpha_i^j+\Delta x_-^i a_i^j),\ \sum_{i=1}^{m}(\Delta x^i\beta_i^j+\Delta x_-^i b_i^j)\right]\right\rangle\\
&=\left\langle\prod_{j=1}^{n}\left[\sum_{i=1}^{m}\Delta x_+^i a_i^j,\ \sum_{i=1}^{m}\Delta x_+^i b_i^j\right],\ \prod_{j=1}^{n}\left[\sum_{i=1}^{m}\Delta x_-^i a_i^j,\sum_{i=1}^{m}\Delta x_-^i b_i^j\right]\right\rangle\\
&=\left\langle\prod_{j=1}^{n}\left[\alpha^j+\sum_{i=1}^{m}\Delta x^i a_i^j,\ \beta^j+\sum_{i=1}^{m}\Delta x^i b_i^j\right],\ \prod_{j=1}^{n}[\alpha^j,\beta^j]\right\rangle\\
&=\left\langle\prod_{j=1}^{n}[\alpha^j+Da^j(x_0)(\Delta x),\ \beta^j+Db^j(x_0)(\Delta x)],\ \prod_{j=1}^{n}[\alpha^j,\beta^j]\right\rangle\end{aligned}$$

式中α^j，β^j是满足下述不等式的任意实数：

$$\alpha^j\leqslant\beta^j \tag{6}$$

$$\alpha^j+Da^j(x_0)(\Delta x)\leqslant\beta^j+Db^j(x_0)(\Delta x) \tag{7}$$

故(3)式成为：

$$D\hat{\Omega}(x_0)(\Delta x)=\left\langle\prod_{j=1}^{n}[\alpha^j+Da^j(x_0)(\Delta x),\beta^j+Db^j(x_0)(\Delta x)],\right.$$
$$\left.\prod_{j=1}^{n}[\alpha^j,\beta^j]\right\rangle \tag{8}$$

引理 1　$\boldsymbol{B}_1(R^n)$在$\boldsymbol{B}(R^n)$中闭。

证明　在$\boldsymbol{B}_1(R^n)$中任取一个收敛点列$\left\{\left\langle\prod_{j=1}^{n}[a_s^j,b_s^j],\prod_{j=1}^{n}\tilde{a}_s^j,\tilde{b}_s^j]\right\rangle\right\}_s$，它是$\boldsymbol{B}(R^n)$

中的一个Cauchy列，从而

$$\left\|\left\langle\prod_{j=1}^{n}[a_s^j,b_s^j],\prod_{j=1}^{n}[\tilde{a}_s^j,\tilde{b}_s^j]\right\rangle-\left\langle\prod_{j=1}^{n}[a_r^j,b_r^j],\prod_{j=1}^{n}[\tilde{a}_r^j,\tilde{b}_r^j]\right\rangle\right\|\to 0$$

$$(r,s\to\infty)$$

$$\Rightarrow\qquad d_H\left(\prod_{j=1}^{n}[a_s^j+\tilde{a}_r^j,\ b_s^j+\tilde{b}_r^j],\ \prod_{j=1}^{n}[\tilde{a}_s^j+a_r^j,\ \tilde{b}_s^j+b_r^j]\right)\to 0$$

$$\Rightarrow\forall j\qquad |a_s^j+\tilde{a}_r^j-\tilde{a}_s^j-a_r^j|\to 0\ (r,s\to\infty)$$

$\Rightarrow\{a_s^j-\tilde{a}_s^j\}_s$是$R^1$中的一个Cauchy列，同理，$\{b_s^j-\tilde{b}_s^j\}$也是$R^1$中Cauchy列，从而$\exists a^j$，$b^j\in R^1$使得

$$a_s^j-\tilde{a}_s^j\to a^j,\ b_s^j-\tilde{b}_s^j\to b^j\ (s\to\infty,\ j=1,2,\cdots,n)$$

$$\Rightarrow\qquad\left\langle\prod_{j=1}^{n}[a_s^j,b_s^j],\prod_{j=1}^{n}[\tilde{a}_s^j,\ \tilde{b}_s^j]\right\rangle\to\left\langle\prod_{j=1}^{n}[\alpha^j+a^j,\beta^j+b^j],\right.$$

$$\left.\prod_{j=1}^{n}[\alpha^j,\ \beta^j]\right\rangle\in\boldsymbol{B}_1(R^n)$$

其中α^j，β^j是满足下述不等式的任意实数：

$$\begin{cases}\alpha^j\leqslant\beta^j\\ \alpha^j+a^j\leqslant\beta^j+b^j\end{cases}$$

$\therefore$ $\boldsymbol{B}_1(R^n)$在$\boldsymbol{B}(R^n)$中闭。 （证毕）

设Ω在$x_0\in R^m$处π—可微，则(1)式成立，并且

$$\frac{\|\hat{\Omega}(x_0+\Delta x)-\hat{\Omega}(x_0)-D\hat{\Omega}(x_0)(\Delta x)\|}{\|\Delta x\|}\to 0\quad(\|\Delta x\|\to 0)$$

故$\forall i$ $(1\leqslant i\leqslant m)$，令$\Delta x=\Delta x^i\cdot e_i$，有

$$\left\|\frac{\hat{\Omega}(x_0+\Delta x)-\hat{\Omega}(x_0)}{\Delta x^i}-\langle A_i(x_0),B_i(x_0)\rangle\right\|$$

$$=\frac{\|\hat{\Omega}(x_0+\Delta x)-\hat{\Omega}(x_0)-D\hat{\Omega}(x_0)(\Delta x)\|}{\|\Delta x\|}\to 0\qquad(\Delta x^i\to 0)$$

$$\Rightarrow\qquad\frac{\hat{\Omega}(x_0+\Delta x)-\hat{\Omega}(x_0)}{\Delta x^i}\to\langle A_i(x_0),\ B_i(x_0)\rangle\ (\Delta x^i\to 0)$$

从而据引理1，$\langle A_i(x_0),B_i(x_0)\rangle\in\boldsymbol{B}_1(R^n)$ $(i=1,2,\cdots,m)$ 由此证得条件(2)是不必要的，即有

推论 若$\Omega:R^m\to\mathscr{B}_1(R^n)$在$x_0$处$\pi$—可微，则 $\forall\Delta x\in R^m$，

$$D\hat{\Omega}(x_0)(\Delta x)\in\boldsymbol{B}_1(R^n)$$

于是，得到下述结论：

定理 1 设给定一个集映$\Omega:G\to\mathscr{B}_1(R^n)$，$G\subset R^m$开，$\Omega(x)=\prod_{j=1}^{n}[a^j(x),\ b^j(x)]$，则

Ω在$x_0\in G$处π—可微$\Leftrightarrow a^j$，b^j $(j=1,2,\cdots,m)$ 都在x_0处可微。

并且当Ω在x_0处π—可微时有(8)式成立。

由于一般来说$D\hat{\Omega}(x_0)$不再是一个集映，即一般地$D\hat{\Omega}(x_0)(\Delta x)\notin\mathscr{B}_1(R^n)$，故要讨论高阶微分，还需要下述命题。

定理 2　给定一个映射$F: G\to \boldsymbol{B}_1(R^n)$，$G\subset R^m$开，$F(x)=\left\langle \prod_{j=1}^{n}[a^j(x),\ b^j(x)],\ \prod_{j=1}^{n}[\tilde{a}^j(x),\ \tilde{b}^j(x)]\right\rangle$，则$F$在$x_0\in G$处可微$\Longleftrightarrow a^j-\tilde{a}^j, b^j-\tilde{b}^j$ $(j=1,2,\cdots,n)$ 都在x_0处可微，并且当F可微时有

$$DF(x_0)(\Delta x)=\left\langle \prod_{j=1}^{n}[\alpha^j+D(a^j-\tilde{a}^j)(x_0)(\Delta x),\ \ \beta^j+D(b^j-\tilde{b}^j)(x_0)(\Delta x)],\ \prod_{j=1}^{n}[\alpha^j,\ \beta^j]\right\rangle$$

式中α^j，β^j是满足下述不等式的任意实数：

$$\alpha^j\leqslant\beta^j \tag{9}$$

$$\alpha^j+D(a^j-\tilde{a}^j)(x_0)(\Delta x)\leqslant\beta^j+D(b^j-\tilde{b}^j)(x_0)(\Delta x) \tag{10}$$

证明　$\Rightarrow$) 设F在$x_0\in G$处可微，由引理1知，

$$\exists\left\langle \prod_{j=1}^{n}[c_i^j,d_i^j],\ \prod_{j=1}^{n}[\tilde{c}_i^j,\ \tilde{d}_i^j]\right\rangle\triangleq\langle A_i(x_0),\ B_i(x_0)\rangle\quad(i=1,2,\cdots,m)$$

使得
$$DF(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i\langle A_i(x_0),\ B_i(x_0)\rangle$$

$$\begin{aligned}&\|F(x_0+\Delta x)-F(x_0)-DF(x_0)(\Delta x)\|\\
&=\Bigg\|\left\langle\prod_{j=1}^{n}[a^j(x_0+\Delta x),\ b^j(x_0+\Delta x)],\ \prod_{j=1}^{n}[\tilde{a}^j(x_0+\Delta x),\ \tilde{b}^j(x_0+\Delta x)]\right\rangle\\
&\quad-\left\langle\prod_{j=1}^{n}[a^j(x_0),\ b^j(x_0)],\ \prod_{j=1}^{n}[\tilde{a}^j(x_0),\ \tilde{b}^j(x_0)]\right\rangle\\
&\quad-\sum_{i=1}^{m}\Delta x^i\left\langle\prod_{j=1}^{n}[c_i^j,\ d_i^j],\ \prod_{j=1}^{n}[\tilde{c}_i^j,\ \tilde{d}_i^j]\right\rangle\Bigg\|\\
&=\max_{1<j<n}\Bigg\{\left|(a^j-\tilde{a}^j)(x_0+\Delta x)-(a^j-\tilde{a}^j)(x_0)-\sum_{i=1}^{m}\Delta x^i(c_i^j-\tilde{c}_i^j)\right|,\\
&\qquad\left|(b^j-\tilde{b}^j)(x_0+\Delta x)-(b^j-\tilde{b}^j)(x_0)-\sum_{i=1}^{m}\Delta x^i(d_i^j-\tilde{d}_i^j)\right|\Bigg\}\end{aligned}$$

从而
$$\frac{\|F(x_0+\Delta x)-F(x_0)-DF(x_0)(\Delta x)\|}{\|\Delta x\|}\to 0\qquad(\|\Delta x\|\to 0)$$

$$\Rightarrow\quad\frac{\left|(a^j-\tilde{a}^j)(x_0+\Delta x)-(a^j-\tilde{a}^j)(x_0)-\sum_{i=1}^{m}\Delta x^i(c_i^j-\tilde{c}_i^j)\right|}{\|\Delta x\|}\to 0\ (\|\Delta x\|\to 0)$$

$\Rightarrow$　$a^j-\tilde{a}^j$在x_0处可微，且$D(a^j-\tilde{a}^j)(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i(c_i^j-\tilde{c}_i^j)$，$j=1,2,\cdots,n$.

同理，$b^j-\tilde{b}^j$在x_0处可微，且$D(b^j-\tilde{b}^j)(x_0)(\Delta x)=\sum_{i=1}^{m}\Delta x^i(d_i^j-\tilde{d}_i^j)$，$j=1,2,\cdots,n$.

$\therefore$　必要性得证。

$\Leftarrow$)　设$a^j-\tilde{a}^j$，$b^j-\tilde{b}^j$，$j=1,2,\cdots,n$都在$x_0\in G$处可微，令$f(x_0): R^m\to\boldsymbol{B}_1(R^n)$为线性映射且

$$f(x_0)(\Delta x)=\left\langle\prod_{j=1}^{n}[\alpha^j+D(a^j-\tilde{a}^j)(x_0)(\Delta x),\ \beta^j+D(b^j-\tilde{b}^j)(x_0)(\Delta x)],\ \prod_{j=1}^{n}[\alpha^j,\ \beta^j]\right\rangle$$

式中α^j，β^j是满足(9)、(10)式的任意实数。

$$\frac{\|F(x_0+\Delta x)-F(x_0)-f(x_0)(\Delta x)\|}{\|\Delta x\|}$$

$$=\frac{1}{\|\Delta x\|}\max\{|(a^j-\tilde{a}^j)(x_0+\Delta x)-(a^j-\tilde{a}^j)(x_0)-D(a^j-\tilde{a}^j)(x_0)(\Delta x)|,$$

$$|(b^j-\tilde{b}^j)(x_0+\Delta x)-(b^j-\tilde{b}^j)(x_0)-D(b^j-\tilde{b}^j)$$

$$\cdot(x_0)(\Delta x)|\}\to 0\qquad(\|\Delta x\|\to 0)$$

$\therefore$　$F(x)$在x_0处可微，并且有$DF(x_0)=f(x_0)$　　　　（证毕）

下面递归地定义$\Omega(x)$的高阶微分。$\mathscr{L}(X,Y)$表示由线性赋范空间X映到Y的线性有界算子全体连同算子范数一起所构成的线性赋范空间。Ω称为在x_0处是二阶π—可微的是指：$D\hat{\Omega}(\cdot):R^m\to\mathscr{L}(R^m,B(R^n))$ 是可微的，即存在线性连续映射$D^2\hat{\Omega}(x_0)$：$R^m\to\mathscr{L}(R^m,B(R^n))$使得

$$D\hat{\Omega}(x)-D\hat{\Omega}(x_0)-D^2\hat{\Omega}(x_0)(x-x_0)=o(\|x-x_0\|)\qquad(11)$$

一般地，Ω被称为在x_0处k阶π—可微是指，$D^{k-1}\hat{\Omega}(x)$在x_0处是可微的。

与[3]中定义7比较，显然这里的可微性条件较弱，[3]中除要求(11)式成立以外，还要求$D^2F(x)$是一个集映。与通常实值函数的微分相仿，在讨论$\hat{\Omega}$的各阶微分时，总是假定自变量增量Δx是保持不变的，在此约定下，我们称$D^2\hat{\Omega}(x_0)(\Delta x)^2$为$\Omega$在$x_0$处的二阶微分。称$D^k\hat{\Omega}(x_0)(\Delta x)^k$为$\Omega$在$x_0$处的$k$阶微分，可以简记为$D^k\hat{\Omega}(x_0)(\Delta x)$。(11) 式相当于

$$\frac{\|D\hat{\Omega}(x)-D\hat{\Omega}(x_0)-D^2\hat{\Omega}(x_0)(x-x_0)\|}{\|x-x_0\|}\to 0\qquad(\|x-x_0\|\to 0)\qquad(12)$$

显然由(12)式可以推出，$\forall\,\Delta x\in R^m$

$$\frac{\|D\hat{\Omega}(x)(\Delta x)-D\hat{\Omega}(x_0)(\Delta x)-D^2\hat{\Omega}(x_0)(x-x_0)(\Delta x)\|}{\|x-x_0\|}\to 0\ (\|x-x_0\|\to 0)\qquad(13)$$

而据据下面的引理2，由(13)式亦可以推出(12)式。

引理 2　设Y是线性赋范空间，X是有限维线性赋范空间，f_n，$f_0\in\mathscr{L}(X,Y)$，则
$\|f_n-f_0\|\to 0\ (n\to\infty)\Longleftrightarrow\forall\,x\in X,\ \|f_n(x)-f_0(x)\|\to 0\ (n\to\infty)$

证明（略）。据引理2，(12)式等价于(13)式，从而有

定理 3　设$\Omega:R^m\to\mathscr{B}_1(R^n)$在$x_0$的某邻域中一阶$\pi$—可微，则$\Omega$在$x_0$处二阶$\pi$—可微$\Longleftrightarrow\forall\,\Delta x\in R^m$，一阶微分$D\hat{\Omega}(\cdot)(\Delta x):R^m\to B(R^n)$是可微的。

对k阶微分亦有类似结论，证明从略。

由定理1、2、3，应用归纳法，便得到了所需的主要定理。

定理 4　给定一个集映$\Omega:G\to\mathscr{B}_1(R^n)$，$G\subset R^m$开，$\Omega(x)=\prod_{j=1}^{n}[a^j(x),\ b^j(x)]$，则

$\Omega(x)$在$x_0\in G$处k阶π—可微$\Longleftrightarrow a^j$，$b^j(j=1,2,\cdots,n)$在x_0处都是k阶可微的。并且此时有

$$D^k\hat{\Omega}(x_0)(\Delta x)=\left\langle\prod_{j=1}^{n}[\alpha^j+D^ka^j(x_0)(\Delta x),\ \beta^j+D^kb^j(x_0)(\Delta x)],\ \prod_{j=1}^{n}[\alpha^j,\ \beta^j]\right\rangle$$

其中α^j，β^j为满足下述不等式的任意实数：

$$\begin{cases}\alpha^j \leqslant \beta^j \\ \alpha^j + D^k a^j(x_0)(\Delta x) \leqslant \beta^j + D^k b^j(x_0)(\Delta x)\end{cases}$$

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On Higher Order Differentials of One Class Multifunctions

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Abstract

In the paper [1], H. T. Banks and M. Q. Jacobs have discussed the one order differential of the multifuntions Ω: $R^m \to \mathscr{B}(R^n)$, Their result has been improved in this paper and the sufficient and necessary conditions for the differentiability are given respectively. The limitation of (3. 2) in [1] is proved to be unnecessary and the sufficient and necessary conditions for the differentialbility of Ω are found. Furthermore, the sufficient and necessary conditions for the higher order differentialbility of Ω are gained and a calculus formula is given.

Key words: multifunction, Hausdorff metric, differential of the multifunction, higher order differential of multifunction