国防科技大学学报
JOURNAL OF NATIONAL UNIVERSITY OF DEFENSE TECHNOLOGY
第12卷第2期 1990年6月 Vol.12 No.2

简支扁壳弯曲问题的一般解

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摘 要 本文建立了四边简支的矩形扁壳弹性弯曲问题的一般解析解。以面内四边位移为零的简支矩形扁壳为例求解了匀布荷载作用下的对称变形解。

关键词 弹性弯曲，扁壳，挠变函数，应力函数
分类号 O342，V414

1 基本方程

矩形扁壳弹性弯曲的基本方程是[1]

$$\frac{1}{Eh}\nabla^2\nabla^2\varphi-\nabla_k^2W=0 \tag{1}$$

$$\nabla_k^2\varphi+D\nabla^2\nabla^2W-P_z=0 \tag{2}$$

式中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$\nabla_k^2=k_y\frac{\partial^2}{\partial x^2}+k_x\frac{\partial^2}{\partial y^2}$$

壳的中面应力为

$$\sigma'_x=-\frac{1}{h}\ \frac{\partial^2\varphi}{\partial y^2},\ \sigma'_y=-\frac{1}{h}\ \frac{\partial^2\varphi}{\partial x^2},\ \tau'_{xy}=\frac{1}{h}\ \frac{\partial^2\varphi}{\partial x\partial y} \tag{3}$$

中面应变为

$$\left.\begin{aligned}\varepsilon'_x&=\frac{\partial u}{\partial x}=-\frac{1}{Eh}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)\\ \varepsilon'_y&=\frac{\partial v}{\partial y}=-\frac{1}{Eh}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\partial^2\varphi}{\partial y^2}\right)\\ \gamma'_{xy}&=\frac{2(1+\mu)}{Eh}\ \frac{\partial^2\varphi}{\partial x\partial y}\end{aligned}\right\} \tag{4}$$

1989年4月20日收稿

底层纤维的弯曲应力为

$$\left.\begin{aligned}\sigma''_x &= -\frac{Eh}{2(1-u^2)}\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\\ \sigma''_y &= -\frac{Eh}{2(1-u^2)}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\\ \tau''_{xy} &= -\frac{Eh}{2(1+\mu)}\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\} \tag{5}$$

符号意义及正负号规定均见文献[1]。取应力函数φ和挠度w为双正弦级数，可以求得四边挠度、弯矩以及中面正应力为零的解。本文将求解面内边界条件为任意情形的简支矩形扁壳弯曲问题的解。

2 挠度函数

四边简支的边界条件是

$$w=0, M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)=0,\ \text{当}x=0, x=a$$

$$w=0, M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)=0,\ \text{当}y=0, y=b$$

取挠度函数为

$$w=\sum_m\sum_n w_{mn}\sin\alpha x\sin\beta y \tag{6}$$

式中 $\alpha=\frac{m\pi}{a}$，$m=1,2,\cdots\cdots$；$\beta=\frac{n\pi}{b}$，$n=1,2,\cdots\cdots$

(6)式满足简支的边界条件。

3 应力函数

将(6)式代入(1)式得

$$\nabla^2\nabla^2\varphi=-Eh\sum_m\sum_n w_{mn}(k_y\alpha^2+k_x\beta^2)\sin\alpha x\sin\beta y \tag{7}$$

令上式的特解为

$$\varphi_1=\sum_m\sum_n\varphi_{mn}\sin\alpha x\sin\beta y \tag{8}$$

将上式代入(7)式，并利用正弦级数的正交性得

$$\varphi_{mn}=-w_{mn}Eh\frac{k_y\alpha^2+k_x\beta^2}{(\alpha^2+\beta^2)^2} \tag{9}$$

(7)式的齐次解可取为[2]

$$\begin{aligned}\varphi_2=&\sum_m[A_m\text{sh}\alpha(b-y)+B_m\text{sh}\alpha y+C_m\alpha(b-y)\text{ch}\alpha(b-y)+\\&D_m\alpha y\text{ch}\alpha y]\sin\alpha x/\text{sh}\alpha b+\sum_n[E_n\text{sh}\beta(a-x)+F_n\text{sh}\beta x+\\&G_n\beta(a-x)\text{ch}\beta(a-x)+H_n\beta x\text{ch}\beta x]\sin\beta y/\text{sh}\beta a+a_{00}+\end{aligned}$$

$$+a_{10}\frac{x}{a}+a_{01}\frac{y}{b}+a_{11}\frac{xy}{ab}+a_{20}\frac{x^2}{a^2}+a_{02}\frac{y^2}{b^2}+a_{21}\frac{x^2y}{a^2b}+a_{12}\frac{xy^2}{ab^2}$$

$$+a_{30}\frac{x^3}{a^3}+a_{03}\frac{y^3}{b^3}+a_{31}\frac{x^3y}{a^3b}+a_{13}\frac{xy^3}{ab^3} \tag{10}$$

a_{00}, a_{10}，a_{01}三项不引起应力，可取消。(7)式的一般解为

$$\varphi=\varphi_1+\varphi_2 \tag{11}$$

将(8)式和(10)式代入(11)式，然后和(6)式一齐代入(2)式可得

$$-k_y\Big\{\sum_m\sum_n\varphi_{mm}\alpha^2\sin\alpha x\sin\beta y+\sum_m[A_m\mathrm{sh}\alpha(b-y)+B_m\mathrm{sh}\alpha y$$

$$+C_m\alpha(b-y)\mathrm{ch}\alpha(b-y)+D_m\alpha y\mathrm{ch}\alpha y]\alpha^2\sin\alpha x/\mathrm{sh}\alpha b-\sum_n[(E_n$$

$$+2G_n)\mathrm{sh}\beta(a-x)+(F_n+2H_n)\mathrm{sh}\beta x+G_n\beta(a-x)\mathrm{ch}\beta(a-x)$$

$$+H_n\beta x\mathrm{ch}\beta x]\beta^2\sin\beta y/\mathrm{sh}\beta a-\left(a_{20}+a_{21}\frac{y}{b}+a_{30}\frac{3x}{a}+a_{31}\frac{3xy}{ab}\right)\frac{2}{a^2}\Big\}$$

$$-k_x\Big\{\sum_m\sum_u\varphi_{mn}\beta^2\sin\alpha x\sin\beta y-\sum_m[(A_m+2C_m)\mathrm{sh}\alpha(b-y)+(B_m$$

$$+2D_m)\mathrm{sh}\alpha y+C_m\alpha(b-y)\mathrm{ch}\alpha(b-y)+D_m\alpha y\mathrm{ch}\alpha y]\alpha^2\sin\alpha x/\mathrm{sh}\alpha b$$

$$+\sum_n[E_n\mathrm{sh}\beta(a-x)+F_n\mathrm{sh}\beta x+G_n\beta(a-x)\mathrm{ch}\beta(a-x)$$

$$+H_n\beta x\mathrm{ch}\beta x]\beta^2\sin\beta y/\mathrm{sh}\beta a-\left(a_{02}+a_{12}\frac{x}{a}+a_{03}\frac{3y}{b}+a_{13}\frac{3xy}{ab}\right)\frac{2}{b^2}\Big\}$$

$$+D\sum_m\sum_n w_{mn}(\alpha^2+\beta^2)^2\sin\alpha x\sin\beta y=p_z \tag{12}$$

将上式中的非正弦级数展成以下的正弦级数

$$1=\sum_m\frac{2(1-\cos m\pi)}{m\pi}\sin\alpha x,\quad \frac{x}{a}=-\sum_m\frac{2\cos m\pi}{m\pi}\sin\alpha x$$

$$\mathrm{sh}\beta(a-x)=\mathrm{sh}\beta a\sum_m\frac{2\alpha}{a(\alpha^2+\beta^2)}\sin\alpha x,\quad \mathrm{sh}\beta x=-\mathrm{sh}\beta a\sum_m\frac{2\alpha\cos m\pi}{a(\alpha^2+\beta^2)}\sin\alpha x$$

$$\beta(a-x)\mathrm{ch}\beta(a-x)=\mathrm{sh}\beta a\sum_m\frac{2\alpha}{a(\alpha^2+\beta^2)}\left(\beta a\mathrm{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\sin\alpha x$$

$$\beta x\mathrm{ch}\beta x=-\mathrm{sh}\beta a\sum_m\frac{2\alpha\cos m\pi}{a(\alpha^2+\beta)}\left(\beta a\mathrm{cth}\beta a-\frac{2\beta^2}{\alpha^2+\beta^2}\right)\sin\alpha x$$

以及相似的等式，则利用正交性得

$$-\varphi_{mn}(k_y\alpha^2+k_x\beta^2)-\{A_mk_y-(A_m+2C_m)k_x-[B_mk_y-(B_m$$

$$+2D_m)k_x]\cos n\pi\}\frac{2\alpha^2\beta}{b(\alpha^2+\beta^2)}-(C_m-D_m\cos n\pi)(k_y-k_x)(\alpha b\mathrm{cth}\alpha b$$

$$-\frac{2\alpha^2}{\alpha^2+\beta^2}\Big)\frac{2\alpha^2\beta}{b(\alpha^2+\beta^2)}-\{E_nk_x-(E_m+2G_n)k_y-[F_nk_x-(F_n$$

$$+2H_n)k_y]\cos m\pi\}\frac{2\alpha\beta^2}{a(\alpha^2+\beta^2)}-(G_n-H_n\cos m\pi)(k_x-k_y)(\beta a\text{cth}\beta a$$

$$-\frac{2\beta^2}{\alpha^2+\beta^2}\Big)\frac{2\alpha\beta^2}{a(\alpha^2+\beta^2)}+8\Bigg[\Big(a_{20}\frac{k_y}{a^2}+a_{02}\frac{k_x}{b^2}\Big)\frac{1-\cos m\pi}{m\pi}\ \frac{1-\cos n\pi}{n\pi}$$

$$-\Big(a_{21}\frac{k_y}{a^2}+a_{03}\frac{3k_x}{b^2}\Big)\frac{1-\cos m\pi}{m\pi}\ \frac{\cos n\pi}{n\pi}-\Big(a_{30}\frac{3k_y}{a^2}+a_{12}\frac{k_x}{b^2}\Big)\frac{\cos m\pi}{m\pi}\ \frac{1-\cos n\pi}{n\pi}$$

$$+3\Big(a_{31}\frac{k_y}{a^2}+a_{13}\frac{k_x}{b^2}\Big)\frac{\cos m\pi}{m\pi}\ \frac{\cos n\pi}{n\pi}\Bigg]+w_{mn}D(\alpha^2+\beta^2)^2=p_{mn} \tag{13}$$

式中
$$p_{mn}=\frac{4}{ab}\int_0^a\int_0^b p_z\sin\alpha x\sin\beta y dxdy \tag{14}$$

4　边界条件

(10)式中的积分常数，可根据面内边界条件来确定。为简单起见，仅考虑对称变形情形。此时m和n仅取奇数值，且

$$A_m=B_m,\ C_m=D_m,\ E_n=F_n,\ G_n=H_n$$

$$a_{11}=a_{21}=a_{12}=a_{30}=a_{03}=a_{31}=a_{13}=0$$

将(11)式代入(4)式，并应用到以上各式，然后积分得

$$Ehu=-\sum_m\sum_n\varphi_{mn}\frac{\beta^2-\mu\alpha^2}{\alpha}\cos\alpha x\sin\beta y+\sum_m\{[A_m(1+\mu)$$

$$+2C_m][\text{sh}\alpha(b-y)+\text{sh}\alpha y]+C_m(1+\mu)[\alpha(b-y)\text{ch}\alpha(b-y)$$

$$+\alpha y\text{ch}\alpha y]\frac{\alpha\cos\alpha x}{\text{sh}\alpha b}-\sum_n\{[E_n(1+\mu)-G_n(1-\mu)][\text{ch}\beta(a-x)$$

$$-\text{ch}\beta x]+G_n(1+u)[\beta(a-x)\text{sh}\beta(a-x)-\beta x\text{sh}\beta x]\frac{\beta\sin\beta y}{\text{sh}\beta a}$$

$$-\Big(a_{02}\frac{2}{b^2}-a_{20}\frac{2\mu}{a^2}\Big)x+f(y) \tag{15}$$

设壳中点的面内位移为零。此时令$x=\frac{a}{2}$，$u=0$，代入上式得

$$f(y)=\Big(\frac{a_{02}}{b^2}-\frac{a_{20}\mu}{a^2}\Big)a$$

对于面内边界位移为零的对称变形问题应有

$$x=0,\ u=0$$
$$y=0,\ u=0$$

将(15)式代入第一个位移条件，并将非正弦级数展成正弦级数，则利用正交性得

$$-\sum_m \varphi_{mn}\frac{\beta^2-\mu\alpha^2}{\alpha}+\sum_m\left[A_m(1+\mu)+2C_m+C_m(1+\mu)\left(\alpha b\,\mathrm{cth}\,\alpha b-\frac{2\alpha^2}{\alpha^2+\beta^2}\right)\right]\frac{4\alpha^2}{a(\alpha^2+\beta^2)}-[E_n(1+\mu)-G_n(1-\mu)]\left(\mathrm{cth}\beta a-\frac{1}{\mathrm{sh}\beta a}\right)\beta-G_n(1+\mu)\beta^2 a=0 \tag{16}$$

将(15)式代入第二个位移条件，并应用到

$$a-2x=a\sum_m\frac{4(1-\cos m\pi)}{m^2\pi^2}\cos\alpha x$$

可得

$$\{A_m(1+\mu)+C_m[2+(1+\mu)\alpha b)\}\alpha+\left(\frac{a_{02}}{b^2}-\frac{a_{20}\mu}{a^2}\right)\frac{8a}{m^2\pi^2}=0 \tag{17}$$

此外，根据整个壳体的平衡应有

$$\sum X=\int_0^b[(N_x)_{x=0}+(N_x)_{x=a}]dy=0$$

将(11)式代入(3)式，然后代入上式得

$$a_{02}=\sum_n(E_n+G_n\beta a\,\mathrm{cth}\beta a)n\pi$$

同样有

$$a_{20}=\sum_m(A_m+C_m\alpha b\,\mathrm{cth}\alpha b)m\pi$$

将以上二式代入(13)，(16)和(17)式，然后分别乘以$\frac{a^2}{k_y}$，a和$\frac{m\pi}{a}$，并应用到(9)式和$\mathrm{cth}\beta a-\frac{1}{\mathrm{sh}\beta a}=\mathrm{th}\frac{\beta a}{2}$可得

$$w_{mn}Ehk_y a^2\left[\left(\frac{m^2+k\lambda^2n^2}{m^2+\lambda^2n^2}\right)^2+\frac{h^2(m^2+\lambda^2n^2)}{12(1-\mu^2)k_y^2}\left(\frac{\pi}{a}\right)^4\right]+\frac{32}{mn\pi}\sum_i\left[A_i+C_i\frac{i\pi}{\lambda}\mathrm{cth}\frac{i\pi}{\lambda}+k\lambda^2(E_i+G_i\lambda i\pi\,\mathrm{cth}\lambda i\pi)\right]i$$
$$+\left\{A_m(k-1)+C_m\left[2k+(k-1)\left(\frac{m\pi}{\lambda}\mathrm{cth}\frac{m\pi}{\lambda}-\frac{2m^2}{m^2+\lambda^2n^2}\right)\right]\right\}\frac{4\lambda_2 m^2n\pi}{m^2+\lambda^2n^2}$$
$$+\left\{E_n(1-k)+G_n\left[2+(1-k)\left(\lambda n\pi\,\mathrm{cth}\lambda n\pi-\frac{2\lambda^2n^2}{m^2+\lambda^2n^2}\right)\right]\right\}\frac{4\lambda^2mn^2\pi}{m^2+\lambda^2n^2}=p_{mn}\frac{a^2}{k_y} \tag{18}$$

$$Ehk_ya^2\sum_m w_{mn}\frac{m^2+k\lambda^2n^2}{(m^2+\lambda^2n^2)^2}\;\frac{\lambda^2n_2-\mu m^2}{m\pi}+4\sum_m A_m m\left[(1+\mu)\frac{\lambda^2n}{m^2+\lambda^2n^2}-\frac{\mu}{n}\right]+4\sum_m C_m m\left\{\left[2+(1+\mu)\left(\frac{m\pi}{\lambda}\mathrm{cth}\frac{m\pi}{\lambda}-\frac{2m^2}{m^2+\lambda^2n^2}\right)\right]\frac{\lambda^2n}{m^2+\lambda^2n^2}\right.$$

$$-\frac{\mu m\pi}{\lambda n}\mathrm{cth}\frac{m\pi}{\lambda}\Big\}-E_n(1+\mu)\lambda n\pi\mathrm{th}\frac{\lambda n\pi}{2}+\frac{4\lambda^2}{n}\sum_m E_m m+G_n\lambda n\pi\Big[(1-\mu)\mathrm{th}\frac{\lambda n\pi}{2}$$

$$+(1+\mu)\lambda n\pi\Big]+\frac{4\lambda^2}{n}\sum_m G_m\lambda m^2\pi\mathrm{cth}\lambda m\pi=0 \qquad (19)$$

$$A_m(1+\mu)+C_m\Big[2+(1+\mu)\frac{m\pi}{\lambda}\Big]-\frac{8\mu}{m^3\pi^2}\sum_n A_n n-\frac{8\mu}{m^2\pi}\sum_n C_n\frac{n^2}{\lambda}\mathrm{cth}\frac{n\pi}{\lambda}$$

$$+\frac{8\lambda^2}{m^3\pi^2}\sum_n E_n n+\frac{8\lambda^2}{m^3\pi}\sum_n G_n\lambda n^2\mathrm{cth}\lambda n\pi=0 \qquad (20)$$

同样有

$$Ehk_y a^2\sum_n w_{mn}\frac{m^2+k\lambda^2n^2}{(m^2+\lambda^2n^2)^2}\ \frac{m^2-\mu\lambda^2n^2}{\lambda^2n\pi}+4\sum_n E_n n\Big[(1+\mu)\frac{m}{m^2+\lambda^2n^2}$$

$$-\frac{\mu}{m}\Big]+4\sum_n G_n n\Big\{\Big[2+(1+\mu)\Big(\lambda n\pi\mathrm{cth}\lambda n\pi-\frac{2\lambda^2n^2}{m^2+\lambda^2n^2}\Big)\Big]\frac{m}{m^2+\lambda^2n^2}$$

$$-\frac{\mu\lambda n\pi}{m}\mathrm{cth}\lambda n\pi\Big\}-A_m(1+\mu)\frac{m\pi}{\lambda}\mathrm{th}\frac{m\pi}{2\lambda}+\frac{4}{\lambda^2m}\sum_n A_n n+C_m\frac{m\pi}{\lambda}\Big[(1$$

$$-\mu)\mathrm{th}\frac{m\pi}{2\lambda}+(1+\mu)\frac{m\pi}{\lambda}\Big]+\frac{4}{\lambda^2m}\sum_n C_n\frac{n^2\pi}{\lambda}\mathrm{cth}\frac{n\pi}{\lambda}=0 \qquad (21)$$

$$E_n(1+\mu)+G_n[2+(1+\mu)\lambda n\pi]-\frac{8\mu}{n^3\pi^2}\sum_m E_m m-\frac{8\mu}{n^3\pi}\sum_m G_n\lambda m^2\mathrm{cth}\lambda m\pi$$

$$+\frac{8}{\lambda^2m^3\pi^2}\sum_m A_m m+\frac{8}{\lambda^3m^3\pi}\sum_m C_m\frac{m^2}{\lambda}\mathrm{cth}\frac{m\pi}{\lambda}=0 \qquad (22)$$

式中 $k=\frac{k_x}{k_y}$, $\lambda=\frac{a}{b}$

由以上五式可以解得 w_{mn}, A_m, C_m, E_n和G_n.

5 算 例

四边位移为零的简支矩形扁壳。$a=b=200$cm，$k_x=k_y=0.005$/cm，$h=2$cm，$E=2.1\times10^6$kg/cm^2，$\mu=0.3$. 匀布荷载作用下 $p_z=25$kg/cm^2. 由(14)式得

$$p_{mn}=\frac{16p_z}{mn\pi^2}$$

当m和n各取4, 8, 12项，求得壳中点的挠度分别是0.2267, 0.2332, 0, 2335cm. 故各取12项已十分精确。由(3)式和(5)式求得$\frac{x}{a}$和$\frac{y}{b}$分别等于0，$\frac{1}{8}$，$\frac{1}{4}$，$\frac{3}{8}$和$\frac{1}{2}$各点的挠度中面应力和底层纤维的弯曲应力见表1.

从表中可以看出，最大的应力是在$\frac{x}{a}=\frac{y}{b}=\frac{1}{8}$上层纤维的压应力$\sigma_x=\sigma_y=$

表 1 扁壳各点的挠度(cm)和应力(kg/cm²)

x/a	y/b	w	σ'_x	σ'_y	σ''_x	σ''_y
0.000	0.000	0.0000	-148	-148	0	0
0.000	0.125	0.0000	142	51	0	0
0.000	0.250	0.0000	22	5	0	0
0.000	0.375	0.0000	-78	-18	0	0
0.000	0.500	0.0000	-98	-29	0	0
0.125	0.125	0.2350	-1197	-1197	1138	1138
0.125	0.250	0.2477	-825	-1762	956	299
0.125	0.375	0.2364	-594	-1930	903	246
0.125	0.500	0.2336	-546	-1969	889	245
0.250	0.250	0.2567	-1352	-1352	-12	-12
0.250	0.375	0.2448	-1002	-1597	-27	-38
0.250	0.500	0.2429	-920	-1668	-21	-20
0.375	0.375	0.2349	-1248	-1248	-36	-36
0.375	0.500	0.2340	-1160	-1329	-27	-14
0.500	0.500	0.2335	-1242	-1242	-6	-6

-2335kg/cm². 除四边外，各点的挠度是相差不多的。由于本例是正方形，也可以改令 $A_m=E_m$，$C_m=G_m$，仅应用(18)～(20)三式即可求解。

对于四边面内应力为零的问题，也可用本文的方法来求解。将(8)式和(10)式代入(11)式再将新(11)式代入(3)式，并令边界上的正应力和剪应力分别等于零，然后将非三角级数展成三角级数，并利用正交性求得相应的控制方程即可解出各积分常数。

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A General Solution for Solving Bending Problem of Simply Supported Rectangular Shallow Shell

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Abstract

A general solution for solving elastic bending problem of simply supported rectangular shallow shells is established in this paper. For example, the symmetric deformation for a simply supported rectangular shallow shell loaded uniformly with the displacement along the four edges being zero, is solved.

Key words: elastic bending; shallow shell; defletion function; stress function