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A Discussion of the Saddle Point Algorithm for Large Scale LP Problems*

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Abstract

This paper discusses an extremely rapid algorithm for lincar programming which is based upon the direct approach to the saddle point of the Lagrangean. The algorithm appears particularly well suited for problems of high dimensions and with a significant numbers of non zero elements.

Key words linear programming; large scale problems; saddle point algorithm

Preface

The author of the paper discovered a new saddlepoint method for solving a special control problem in 1980. Based on this idea, in 1985, the author developed a new method which could solve large scale LP problems. The new method was entitled; "The LP Saddlepoint Algorithm for LP" and was presented at the "ORSA/TIMS-St. Louis National Meeting" that same year. In August 1988, the author presented another paper entitled; "A New Algorithm for LP Based on Direct Saddle point Convergence"at the"13th International Symposium on Mathematical Programming"in Tokyo, Japan. In 1990, the author and his assistants developed the algorithm computer software and established the first LP Saddle Point Algorithm Research Center in China. In 1991, the software was applied to production management problems in large scale iron and steel plants and oil refineries, which brought about several millions of dollars in profits. The software can solve LP problems of over 10,000 dimensions on a 486 CPU computer.

This paper discusses: (1) saddle point sufficient and necessary conditions for LP problems; (2) the iterative formula; (3) convergence of the algorithm; and (4) computational results.

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1 Saddle Point Sufficient and Necessary Conditions for LP Problems

Consider the general LP problems in the form:

(P) $\max \ c^T X \quad \text{s.t.} \ AX + b \geqslant 0, \quad X \geqslant 0$ (1)

with dual

(D) $\min \ b^T \lambda \quad \text{s.t.} \ A^T\lambda + c \leqslant 0, \quad \lambda \geqslant 0$ (2)

and Lagrangean

$$Z(X,\lambda) = c^T X + \lambda^T (AX + b) \qquad X \geqslant 0, \quad \lambda \geqslant 0 \tag{3}$$

Definition. A point (X^*, λ^*) with $X^* \geqslant 0$, $\lambda^* \geqslant 0$ is said to be a saddle point for $Z(X,\lambda)$ if it satisfies:

$$Z(X^*, \lambda^*) \geqslant Z(X, \lambda^*) \quad \text{for all} \quad X \geqslant 0 \tag{4}$$

$$Z(X^*, \lambda^*) \leqslant Z(X^*, \lambda) \quad \text{for all} \quad \lambda \geqslant 0 \tag{5}$$

Theorem 1. Let $X^* \geqslant 0$ and $\lambda^* \geqslant 0$. If (X^*, λ^*) is a saddle point for $Z(X,\lambda)$ then

(a) $AX^* + b \geqslant 0$ (6)

(b) $A^T\lambda^* + c \leqslant 0$ (7)

(c) $X^{*T}(A^T\lambda^* + c) = 0$ (8)

(d) $\lambda^{*T}(AX^* + b) = 0$ (9)

Proof. (1) **By formula (5),(3),**

$$c^T X^* + \lambda^{*T}(AX^* + b) \leqslant c^T X^* + \lambda^T (AX^* + b)) \quad \text{for all} \quad \lambda \geqslant 0 \tag{10}$$

or

$$(\lambda^{*T} - \lambda^T)(AX^* + b) \leqslant 0 \qquad \text{for all} \quad \lambda \geqslant 0 \tag{11}$$

If $AX^* + b < 0$, λ may be chosen sufficiently large so that (11) is violated. Thus (a) must hold.

(2) **By formula (4),(3),**

$$c^T X^* + \lambda^{*T}(AX^* + b) \geqslant c^T X + \lambda^{*T}(AX + b) \quad \text{for all} \quad X \geqslant 0 \tag{12}$$

or

$$(X^{*T} - X^T)(A^T\lambda^* + c) \geqslant 0 \qquad \text{for all} \quad X \geqslant 0 \tag{13}$$

If $A^T\lambda^* + c > 0$, X may be chosen sufficiently large so that (13) is violated. Thus (b) must hold.

(3) If $X = 0$, (13) yields

$X^{*T}(A^T\lambda^* + c) \geqslant 0$ (14)

But $X^* \geqslant 0$ and $(A^T\lambda^* + c) \leqslant 0$ imply

$X^{*T}(A^T\lambda^* + c) \leqslant 0$ (15)

Thus (c) must hold.

(4) If $\lambda = 0$, (11) yields

$\lambda^{*T}(AX^* + b) \leqslant 0$ (16)

But $\lambda^* \geqslant 0$ and $(AX^* + b) \geqslant 0$ imply

$\lambda^{*T}(AX^* + b) \geqslant 0$ (17)

Thus (d) must hold.

Theorem 2. Let $X^* \geqslant 0$ and $\lambda^* \geqslant 0$. Then (X^*, λ^*) is a saddle point for $Z(X, \lambda)$ if (a), (b), (c), and (d) hold.

Proof. **By (c),**

$Z(X^*, \lambda^*) = \lambda^{*T} b$ (18)

By Definition,

$Z(X, \lambda^*) = \lambda^{*T} b + X^T(A^T\lambda^* + c)$ (19)

Since $(A^T\lambda^* + c) \leqslant 0$ then

$Z(X, \lambda^*) \leqslant \lambda^{*T} b = Z(X^*, \lambda^*)$ for all $X \geqslant 0$ (20)

By (d),

$Z(X^*, \lambda^*) = c^T X^*$ (21)

By Definition,

$Z(X^*, \lambda) = c^T X^* + \lambda^T(AX^* + b)$ (22)

Since $(AX^* + b) \geqslant 0$ then

$Z(X^*, \lambda) \geqslant c^T X^* = Z(X^*, \lambda^*)$ for all $\lambda \geqslant 0$ (23)

Thus the (X^*, λ^*) must be a saddle point.

Theorem 3. If (X^*, λ^*) is a saddle point for $Z(X, \lambda)$, then X^* solves the primal problem (P), λ^* solves the dual problem (D).

Proof, Since (X^*, λ^*) is a saddle point the conditions of Theorem 1 must hold.

By definition of the saddle point

$c^T X^* + \lambda^*(AX^* + b) \geqslant c^T X + \lambda^{*T}(AX + b)$ for all $X \geqslant 0$ (24)

By (d) in Theorem 1,

$c^T X^* \geqslant c^T X + \lambda^*(AX + b)$ for all $X \geqslant 0$ (25)

If X is a feasible solution for the problem (P), it satisfies the constraints of (P), so that

$AX + b \geqslant 0$, $\lambda^{*T}(AX + b) \geqslant 0$ then

$c^T X^* \geqslant c^T X$, X is a feasible solution of (P), X^* solves the primal problem (P).

By definition,

$b^T\lambda^* + X^{*T}(A^T\lambda^* + c) \leqslant b^T\lambda + X^{*T}(A^T\lambda + c)$ for all $\lambda \geqslant 0$ (26)

By (c) in Theorem 1,

$$b^T\lambda^* \leqslant b^T\lambda + X^{*T}(A^T\lambda + c) \qquad \text{for all} \quad \lambda \geqslant 0 \tag{27}$$

If λ is a feasible solution for the problem (D), it satisfies the constraints of (D), so that

$$A^T\lambda + c \leqslant 0, \quad X^{*T}(A^T\lambda + c) \leqslant 0 \text{ then}$$

$b^T\lambda^* \leqslant b^T\lambda$, λ is a feasible solution of (D), λ^* solves the dual problem (D).

Theorem 4. If (X^*, λ^*) is a saddle point for $Z(X,\lambda)$ then $Z(X^*, \lambda^*) = c^T X^* = b^T\lambda^*$.

Proof. **By definition.**

$$Z(X^*, \lambda^*) = c^T X^* + \lambda^{*T}(AX^* + b)$$

$$Z(X^*, \lambda^*) = b^T\lambda^* + X^{*T}(A^T\lambda^* + c)$$

By (c) and (d) in Theorem 1

$$Z(X^*, \lambda^*) = c^T X^* = b^T\lambda^* \tag{28}$$

We consider the following LP problems in the standard form:

(P′) $\quad \max \ c^T X \quad \text{s.t.} \quad AX + b = 0, \quad X \geqslant 0$ $\qquad (29)$

with dual

(D′) $\quad \min \ b^T\lambda \quad \text{s.t.} \quad A^T\lambda + c \leqslant 0, \quad \lambda$ unrestricted $\qquad (30)$

and Lagrangean

$$Z'(X,\lambda) = c^T X + \lambda^T(AX + b) \qquad X \geqslant 0, \ \lambda \text{ unrestricted}$$

Theorem 5. Let $X^* \geqslant 0$, λ^* (unrestricted). If (X^*, λ^*) is a saddle point of $Z'(X,\lambda)$, if and only if

(a′) $AX^* + b = 0$ $\qquad (31)$

(b′) $A^T\lambda^* + c \leqslant 0$ $\qquad (32)$

(c′) $X^{*T}(A^T\lambda^* + c) = 0$ $\qquad (33)$

X^* solves the problem (P'), λ^* solves the problem (D'). The proof of Theorem 5 is omitted.

2 Application of the Principle to the LP Lagrangean

Consider the general LP problem in the form:

$$\max c^T X \quad \text{s.t.} \ AX + b = 0 \qquad X \geqslant 0 \tag{39}$$

with dual

$$\min b^T\lambda \quad \text{s.t.} \ A^T\lambda + c \leqslant 0, \qquad \lambda \text{ unrestricted} \tag{40}$$

where A is a matrix of order $m \times n$,

and Lagrangean

$$Z(X,\lambda) = c^T X + \lambda^T(AX + b)$$

Setting: $X(t) = (\chi_1(t), \chi_2(t), \cdots, \chi_i(t), \cdots, \chi_n(t))$

$X_l(t) = (\chi_{11}(t), \chi_{12}(t), \cdots, \chi_{li}(t), \cdots, \chi_{ln}(t))$

$\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_i(t), \cdots, \lambda_m(t))$

$\lambda_l(t) = (\lambda_{11}(t), \lambda_{12}(t), \cdots, \lambda_{li}(t), \cdots, \lambda_{lm}(t))$

Consider the iterative process given as follows where positive scalars ρ_1 and ρ_2 are to be specified later:

$$\begin{cases} \lambda_I(t) = \lambda(t) - \rho_1(AX(t) + b) \\ X_I(t) = X(t) + \rho_1(A^T\lambda(t) + c) \\ \chi_{Ii}(t) = 0, \text{ if } \chi_{Ii}(t) < 0 \qquad (i = 1,2,\cdots,n) \\ \lambda(t+1) = \lambda(t) - \rho_2(AX_I(t) + b) \\ X(t+1) = X(t) + \rho_2(A^T\lambda_I(t) + c) \\ \chi_i(t+1) = 0, \text{ if } \chi_{Ii}(t) = 0 \qquad (i = 1,2,\cdots,n) \end{cases} \tag{41}$$

It is noted that formula (41) can be replaced by formula (41′):

$$\begin{cases} \lambda_I(t) = \lambda(t) - \rho_1(AX(t) + b) \\ X_I(t) = X(t) + \rho_1(A^T\lambda(t) + c) \\ \chi_{Ii}(t) = 0, \text{ if } \chi_{Ii}(t) < 0 \\ \lambda(t+1) = \lambda(t) - \rho_2(AX_I(t) + b) \\ X(t+1) = X(t) + \rho_2(A^T\lambda_I(t) + c) \\ \chi_i(t+1) = 0, \text{ if } \chi_{Ii}(t+1) < 0 \end{cases} \tag{41'}$$

Suppose the set in formula (42):

$$\begin{matrix} B(t) = \{i \mid \chi_{Ii}(t) > 0\} \\ C(t) = \{i \mid \chi_{Ii}(t) = 0\} \end{matrix} \qquad (i = 1,2,\cdots,n) \tag{42}$$

By (41) and (42),

$$\begin{cases} \lambda_I(t) = \lambda(t) - \rho_1(A_BX_B(t) + b) \\ X_{IB}(t) = X_B(t) + \rho_1(A_B^T\lambda(t) + c_B) \\ \lambda(t+1) = \lambda(t) - \rho_2(A_BX_{IB}(t) + b) \\ X_B(t+1) = X_B(t) + \rho_2(A_B^T\lambda_1(t) + c_B) \end{cases} \tag{43}$$

By (43),

$$\begin{cases} A_BX_B(t+1) + b = (I - \rho_1\rho_2A_BA_B^T)(A_BX_B(t) + b) + \rho_2A_B(A_B^T\lambda(t) + c_B) \\ A_B^T\lambda(t+1) + c_B = (I - \rho_1\rho_2A_B^TA_B)(A_B^T\lambda(t) + c_B) - \rho_2A_B^T(A_BX_B(t) + b) \end{cases} \tag{44}$$

Define "error" vectors $R(t)$ and $\gamma(t)$:

$$R(t) = \begin{bmatrix} A_BX_B(t) + b \\ A_B^T\lambda(t) + c_B \end{bmatrix} \tag{45}$$

$$\gamma(t) = \begin{bmatrix} A_B^T & (A_BX_B(t) + b) \\ A_B & (A_B^T\lambda(t) + c_B) \end{bmatrix} = \begin{bmatrix} A_B^T & 0 \\ 0 & A_B \end{bmatrix} R(t) \tag{46}$$

By (44),

$$R(t+1) = M_1 \cdot R(t), \quad M_1 = \begin{bmatrix} I - \rho_1\rho_2A_BA_B^T & \rho_2A_B \\ -\rho_2A_B^T & I - \rho_1\rho_2A_B^TA_B \end{bmatrix} \tag{47}$$

Suppose $g_i (i=1,\cdots,k)$, $k \leqslant m$ are the characteristic roots of matrices $A_BA_B^T$ and $A_B^TA_B$.

Let π_1, π_2 be the orthogonal matrices and

$$\pi_1 A_B A_B^T \pi_1^T = \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_1 = \begin{pmatrix} g_1 & & & & 0 \\ & \ddots & & & \\ & & g_i & & \\ & & & \ddots & \\ 0 & & & & g_k \end{pmatrix} \tag{48}$$

$$\pi_2^T A_B^T A_B \pi_2 = \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} g_1 & & & & 0 \\ & \ddots & & & \\ & & g_i & & \\ & & & \ddots & \\ 0 & & & & g_k \end{pmatrix} \tag{49}$$

Theorem 6. For iteration formula (43), if $\alpha = \max_i((1-\rho_1\rho_2 g_i)^2 + \rho_2^2 g_i)$ then $\|\gamma(t+1)\| \leqslant \alpha \|\gamma(t)\|$, where α is a scalar.

Proof. **By (46),**

$$\|\gamma(t+1)\|^2 = R^T(t+1) \begin{pmatrix} A_B & 0 \\ 0 & A_B^T \end{pmatrix} \begin{pmatrix} A_B^T & 0 \\ 0 & A_B \end{pmatrix} R(t+1) \tag{50}$$

By (47) and (50),

$$\|\gamma(t+1)\|^2 = R^T(t) M_1^T \begin{pmatrix} A_B A_B^T & 0 \\ 0 & A_B^T A_B \end{pmatrix} M_1 R(t)$$

$$= R^T(t) \begin{pmatrix} I - \rho_1\rho_2 A_B A_B^T & -\rho_2 A_B \\ \rho_2 A_B^T & I - \rho_1\rho_2 A_B^T A_B \end{pmatrix} \begin{pmatrix} A_B A_B^T & 0 \\ 0 & A_B^T A_B \end{pmatrix}$$

$$\begin{pmatrix} I - \rho_1\rho_2 A_B A_B^T & \rho_2 A_B \\ -\rho_2 A_B^T & I - \rho_1\rho_2 A_B^T A_B \end{pmatrix} R(t) = R^T(t) \begin{pmatrix} A_B A_B^T & 0 \\ 0 & A_B^T A_B \end{pmatrix}$$

$$\begin{pmatrix} (I - \rho_1\rho_2 A_B A_B^T)^2 + \rho_2^2 A_B A_B^T & 0 \\ 0 & (I - \rho_1\rho_2 A_B^T A_B)^2 + \rho_2^2 A_B^T A_B \end{pmatrix} R(t) \tag{51}$$

By (48) and (49),

$$\begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2^T \end{pmatrix} \begin{pmatrix} A_B A_B^T & 0 \\ 0 & A_B^T A_B \end{pmatrix} \begin{pmatrix} \pi_1^T & 0 \\ 0 & \pi_2 \end{pmatrix} = \begin{pmatrix} G_1 & & & 0 \\ & 0 & & \\ & & G_2 & \\ 0 & & & 0 \end{pmatrix} \tag{52}$$

By (51) and (52),

$$\|\gamma(t+1)\|^2 = R^T(t) \begin{pmatrix} \pi_1^T & 0 \\ 0 & \pi_2 \end{pmatrix} \begin{pmatrix} G_1 & & & 0 \\ & 0 & & \\ & & G_2 & \\ 0 & & & 0 \end{pmatrix} M \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2^T \end{pmatrix} R(t) \tag{53}$$

$$M = \begin{pmatrix} \left(\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} - \rho_1\rho_2 \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} \right)^2 + \rho_2^2 \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \left(\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \rho_1\rho_2 \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} \right)^2 + \rho_2^2 \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{54}$$

Suppose α is the greatest characteristic root of matrix M:

$$\alpha = \max_i((1 - \rho_1\rho_2 g_i)^2 + \rho_2^2 g_i) \qquad (i = 1,2,\cdots,k) \tag{55}$$

By (53), (54), and (55),

$$\|\gamma(t+1)\|^2 \leqslant \alpha R^T(t) \begin{bmatrix} \pi_1^T & 0 \\ 0 & \pi_2 \end{bmatrix} \begin{bmatrix} G_1 & & 0 \\ & 0 & \\ & & G_2 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2^T \end{bmatrix} R(t) \tag{56}$$

By (50),

$$\|\gamma(t+1)\|^2 = R^T(t) \begin{bmatrix} A_B A_B^T & 0 \\ 0 & A_B^T A_B \end{bmatrix} R(t)$$

$$= R^T(t) \begin{bmatrix} \pi_1^T & 0 \\ 0 & \pi_2 \end{bmatrix} \begin{bmatrix} G_1 & & 0 \\ & 0 & \\ & & G_2 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2^T \end{bmatrix} R(t) \tag{57}$$

By (56) and (57),

$$\|\gamma(t+1)\|^2 \leqslant \alpha \|\gamma(t)\|^2 \tag{58}$$

Thus the theorem is proved.

Theorem 7. If $\rho_1 = \dfrac{k}{\sqrt{\mu g_{\max}}}$, $\rho_2 = \dfrac{2}{k^2+1}\rho_1 \quad \mu > 1$,

$g_{\max} = \max_i\{g_i\}$, $g_i > 0$ then

$$0 < (1 - \rho_1\rho_2 g_i)^2 + \rho_2^2 g_i < 1 \qquad (i = 1,2,\cdots,k)$$

Proof. Suppose $S = (1 - \rho_1\rho_2 g_i)^2 + \rho_2^2 g_i$ (59)

$$\frac{ds}{dg_i} = (\rho_2^2 - 2\rho_1\rho_2) + 2\rho_1^2\rho_2^2 g_i$$

$$\text{if } g_i = \frac{2\rho_1 - \rho_2}{2\rho_1^2\rho_2} \qquad \frac{ds}{dg_i} = 0, \tag{60}$$

$$\frac{d^2 s}{dg_i^2} = 2\rho_1^2\rho_2^2 > 0$$

So that S has a minimal value Smin if $g_1 = g^* = \dfrac{2\rho_1 - \rho_2}{2\rho_1^2\rho_2}$.

Because $\rho_1 = \dfrac{k}{\sqrt{\mu g_{\max}}}$, $\rho_2 = \dfrac{2}{k^2+1}\dfrac{k}{\sqrt{\mu g_{\max}}}$,

$$g^* = \frac{2\rho_1 - \rho_2}{2\rho_1^2\rho_2} = \frac{\mu g_{\max}}{2} \tag{61}$$

$$\text{Smin} = (1 - \rho_1\rho_2 g^*)^2 + \rho_2^2 g^* = 1 - \frac{k^4}{(k^2+1)^2} \tag{62}$$

so that 0<Smin<1.

By (60),

$$S = 1, \quad \text{if } g_i = 0$$

$$S = 1, \quad \text{if } g_i = \mu g_{max}$$

$S(g_i)$ is a quadratic curve as shown in Figure 1.

So that $0<(1-\rho_1\rho_2 g_i)^2+\rho_2^2 g_i<1$, if $0<g_i\leqslant g_{max}$.

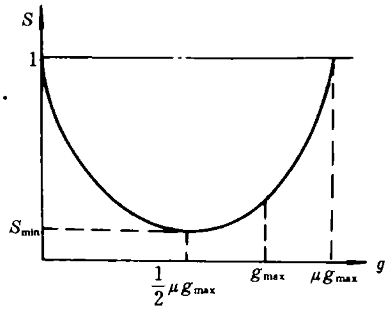

Figure 1

Theorem 8. If $X_B(t)=X_B^*$, $\lambda(t)=\lambda^*$ and

$$\gamma^*(t) = \begin{bmatrix} A_B^T & 0 \\ 0 & A_B \end{bmatrix} R^*(t)$$

$$= \begin{bmatrix} A_B^T & 0 \\ 0 & A_B \end{bmatrix} \begin{bmatrix} A_B X_B^* + b \\ A_B^T\lambda^* + c_B \end{bmatrix} = 0$$

then $R^*(t)$ is a minimum variance.

Proof. By (45) and the theorem condition,

$$\left(\frac{d\|R(t)\|^2}{dX_B(t)}\right)_{X_B(t)=X_B^*} = 2A_B^T(A_B X_B^* + b) = 0 \tag{63}$$

$$\left(\frac{d\|R(t)\|^2}{d\lambda(t)}\right)_{\lambda(t)=\lambda^*} = 2A_B(A_B^T\lambda^* + c_B) = 0 \tag{64}$$

and

$$\frac{d^2\|R(t)\|^2}{dX_B(t)^2} = 2A_B^T A_B \tag{65}$$

$$\frac{d^2\|R(t)\|^2}{d\lambda(t)^2} = 2A_B A_B^T \tag{66}$$

Because the matrices $A_B^T A_B$ and $A_B A_B^T$ are the nonnegative definite and by (63), (64), (65), and (66) $\|R(t)\|^2$ has a minimum if $X_B(t)=X_B^*$ and $\lambda(t)=\lambda^*$. So that $R(t)$ is a minimum variance.

Theorem 9. For formula (43), if $\rho_1=\dfrac{k}{\sqrt{\mu g_{max}}}$, $\rho_2=\dfrac{2}{k^2+1}\rho_1$ $\mu>1$,

then $\|\gamma(t)\|^2 \underset{t\to\infty}{\to} \|\gamma(\infty)\|^2=0$.

Proof. **By Theorem 6,** $\|\gamma(t+1)\|^2\leqslant\alpha\|\gamma(t)\|^2$

By Theorem 7, $0<\alpha<1$

So that $t\to\infty$ $\|\gamma(t)\|^2\to\|\gamma(\infty)\|^2=0$

Theorem 10. For formula (43), if $\rho_1=\dfrac{k}{\sqrt{\mu g_{max}}}$, $\rho_2=\dfrac{2}{k^2+1}\rho_1$ $\mu>1$

and the two equations $A_B X_B(t)+b=0$ and $A_B^T\lambda(t)+c_B=0$ have solutions, then as $t\to\infty$, $X(t)\to X(\infty)$ and $\lambda(t)\to\lambda(\infty)$. $X(\infty)$ must be the solution of (P) and $\lambda(t)$ must be the solution of (D).

Proof. **By Theorem 9,**

$$\|\gamma(\infty)\|=0 \tag{67}$$

By (67) and Theorem 8, the two equations have solutions,

$$A_B X_B(\infty)+b=0 \tag{68}$$

$$A_B^T\lambda(\infty)+c_B=0 \tag{69}$$

By (42),

$$X_{IB}(\infty)>0,\ X_{IC}(\infty)=0 \tag{70}$$

By (41),

$$X_C(\infty)=0,\ \text{if } X_{IC}(\infty)=0 \tag{71}$$

By (43) and (69),

$$X_B(\infty)=X_{IB}(\infty)>0 \tag{72}$$

By (71) and (72),

$$X(\infty)=\begin{bmatrix} X_B(\infty) \\ X_C(\infty) \end{bmatrix}\geqslant 0 \tag{73}$$

By (41) and (70),

$$X_C(\infty)+\rho_1(A_C^T\lambda(\infty)+c_C)\leqslant 0 \tag{74}$$

By (71),

$X_C(\infty)=0$ and $\rho_1>0$, so that

$$A_C\lambda(\infty)+c_C\leqslant 0 \tag{75}$$

By (69) and (75),

$$A^T\lambda(\infty)+c=\begin{bmatrix} A_B \\ A_C \end{bmatrix}^T\lambda(\infty)+\begin{bmatrix} c_B \\ c_C \end{bmatrix}\leqslant 0 \tag{76}$$

Because $$X^T(\infty)(A^T\lambda(\infty)+c)=\begin{bmatrix} X_B(\infty) \\ X_C(\infty) \end{bmatrix}\left(\begin{bmatrix} A_B^T \\ A_C^T \end{bmatrix}\lambda(\infty)+\begin{bmatrix} c_B \\ c_C \end{bmatrix}\right) \tag{77}$$

By (69), (71) and (77),

$$X^T(\infty)(A^T\lambda(\infty)+c)=0 \tag{78}$$

Because $$A_X(\infty)+b=(A_BA_C)\begin{bmatrix} X_B(\infty) \\ X_C(\infty) \end{bmatrix}+b \tag{79}$$

By (71) and (68),

$$AX(\infty)+b=0 \tag{80}$$

By (73), (80), (76), and (78),

$$\begin{cases} X(\infty)\geqslant 0 \\ AX(\infty)+b=0 \\ A^T\lambda(\infty)+c\leqslant 0 \\ X^T(\infty)(A^T\lambda(\infty)+c)=0 \end{cases} \tag{81}$$

By theorem 5, the point $(X(\infty),\lambda(\infty))$ is a saddle point, so that $X(\infty)$ is the optimal solution of (P'), $\lambda(\infty)$ is the optimal solution of (D').

3 Some Computational Results

AT&T Bell Lab provided fifty (50) LP problems (the NET LIB) for testing purposes. Table 1 (restrict to length, deleted) shows the comparison of the problem solutions between the NET LIB and the LP Saddle Point Algorithm. It may be noted that the objective function values differ in some instances, however, all the solutions of the Saddle

Point Algorithm were strictly examined. We improved the NET LIB problems into the form of formula (29). If X^* is the primal problem solution, λ^* is the dual solution, then they were checked to satisfy the following conditions:

(1) $X^* = \begin{bmatrix} X_B^* \\ X_C^* \end{bmatrix}$, $X_B^* > 0 \quad X_C^* = 0$

(2) $\|A_B X_B^* + b\| \leqslant 10^{-5}$

(3) $\|A_B^T \lambda^* + c_B\| \leqslant 10^{-4}$

(4) $A_C^T \lambda^* + c_c \leqslant 10^{-4}$

(5) $\|c_B^T X_B^* - b^T \lambda^*\| \leqslant 10^{-4}$

It is obvious that (X^*, λ^*) satisfies the saddle point's sufficient and necessary condition which is shown in Theorem 5. Thus, X^* and λ^* must be the optimal solutions of the problems (P') and (D').

The name GREENBEA was the only problem for which convergence could not be achieved. A careful examination of the computer software proved that the model data of the problem contained several errors.

The Saddle Point Algorithm's results were obtained on a UNISYS U6000/35, CPU80486 system with memory of 16MB and the speed of 26 MIPS while utilizing Algorithmic Language FORTRAN77.

4 Significant Characteristics of the New Algorithm

(a) Selection of an initial point is arbitrary and no special start-up procedures are required.

(b) The number of iterations is independent of the dimensions of the LP problems. It depends on gmin/gmax. gmax is the greatest characteristic root of $A_B^T A_B$ while gmin is the smallest.

(c) Each of the iterations is a vector times a matrix, therefore the computational results prove extremely fast.

(d) It is not necessary to solve a system of linear equations during the entire computational process.

(e) The solutions to both primal and dual problems approach the saddlepoint simultaneously. The stop conditions of computer are the saddlepoint's sufficient and necessary condition, therefore the precision of the solutions will be guaranteed.

(f) The essential matrix A is not changed during the iterative process. Cumulative errors from the transformation of the matrix do not occur.

(g) Due to the above fact, matrix A is not changed in the computational process and its density is kept unchanged, therefore, the computer memory will be saved.

(h) Degeneracy presents no problem for the algorithm.

(i) The algorithm is parallel and suitable for a multi-CPU super computer.

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关于大型线性规划问题鞍点算法的讨论

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