$$|f(x_j)-f(y_j)| \leqslant \frac{2\lambda_n^2 \mathrm{dia}^2(D')}{\left[\frac{2}{3}R\right]^{(KM)^{1/1-n}} \mathrm{dia}(f(A))} \left[\frac{1}{4}R\right]^{(KM)^{1/1-n}-\alpha} |x_j-y_j|^\alpha$$

(b′) $\gamma_j \not\subset \overline{D}$. Then $\gamma_j \cap \partial D \neq \varnothing$ and there exist at least two points of $\gamma_j \cap \partial D$. Let x_j' and y_j' be the nearest points from x_j and y_j respectively. Denote by ξ_j and η_j the open segments joining x_j to x_j' and y_j to y_j' respectively, then $\xi_j \subset \overline{D}$ and conclude that there exists constant $M'>0$ satisfying

$$|f(x_j)-f(y_j)| \leqslant M'|x_j-y_j|^\alpha \tag{3.12}$$

If $|x_j-y_j| \geqslant \frac{R}{4}$, then by the boundness of D' we have

$$|f(x_j)-f(y_j)| \leqslant \frac{\mathrm{dia}(D')}{(\frac{1}{4}R)^\alpha}|x_j-y_j|^\alpha \tag{3.13}$$

According to the above discuss, we conclude that there exists constand $M>0$ satisfying $|f(x_j)-f(y_j)|/|x_j-y_j|^\alpha \leqslant M$ for sufficient large j, this contradicts with $|f(x_j)-f(y_j)|/|x_j-y_j|^\alpha \to \infty$. Thus $f \in Lip_\alpha(D)$.

We wish to thank professor A. N. Fang and professor J. M. Wu for their encouragement to write this paper.

References

1 Gehring F W. Martio O. Lipschitz classes and quasiconformal mappings. Ann Acad Sci Fenn Ser Math, 1988, 10: 203～219

2 Gehring F W, Hayman W K, Hinkkanen A. Analytic functions satisfying Hölder conditions on the boundary. J Approx Theory, 1982, 35: 353～361

3 Väisälä J. Lectures on n－dimensional quasiconformal mappings. Lecture Notes in Math, Springer－Verlag, Berlin and New York, 1971, 229

4 Gehring F. W. Rings and quasiconformal mappings in space. Trans Amer Math Soc, 1962, 103

5 Vuorinen M. Conformal geometry and quasiregular mappings. Lecture Notes in Math. Springer－Verlag, Berlin and New york, 1988, 1319

6 Gehring F W. Symmetrization of rings in space. Trans Amer Math. Soc 1961, 101: 499～519

拟共形映照和 Hölder 连续性

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摘　要　设 f 是 R^n 中的域 D 到有界的 $M-QED$ 域上的 K－拟共形映照，$0<\alpha\leqslant(KM)^{1/1-n}$. 在本文中作者证明了 $f\in Lpi_\alpha(\partial D)$ 的充要条件是 $f\in Lpi_\alpha(D)$。

关键词　K－拟共形映照，模，$M-QED$ 域，*Hölder* 连续

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Quasiconformal Mappings and Hölder Continuity*

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Abstract Let f be a K−quasiconformal mapping which maps $D\subset R^n$ onto bounded M−QED domain $D'\subset R^n$, $0<\alpha\leqslant (KM)^{1/1-n}$. In this paper, the authors proved that $f\in Lip_\alpha$ (∂D) if and only if $f\in Lip_\alpha$ (D).

Key words K−quasiconformal mappings, module, M−QED domain, Hölder continuity

1 Introduction

In this paper, we shall adopt the standard notation $f\in Lip_\alpha$ (A) and definition M−QED domain in [1].

F. W. Gehring, W. K. Hayman and A. Hinnkanen established the following theorem in [2]:

THEOREM A Suppose that D, D' are Jordan domains in $\overline{R}^2$, f is a conformal mapping which maps D onto D', $0<\alpha\leqslant 1$, If $f\in Lip_\alpha$ (∂D), then $f\in Lip_\alpha$ (D).

In this paper, we shall extend the above result to quasiconformal mapping and obtain the following result:

THEOREM 1 Suppose that $D\subset R^n$ is a domain, $D'\subset R^n$ is a bounded M−QED domain, f is a K−quasiconformal mapping which maps D onto D', $0<\alpha\leqslant (KM)^{1/1-n}$, If $f\in Lip_\alpha$ (∂D), then $f\in Lip_\alpha$ (D).

2 Preliminary knowledge

we shall adopt the relatively standard notation and terminology of [3]. Unit vectors in the directions of the rectangular coordinate axes in R^n are denote by $e_1, e_2, \cdots, e_n$. For $x\in R^n$ and $r>0$, we let $B^n(x, r) = \{z\in R^n: |z-x|<r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r)=B^n(0,r)$, $B^n=B^n(1)$. We follow. J. Väisälä' s definition of K−quasiconformality [3] which is also equivalent to $K^{1/n-1}$−quasiconformality in the definition given

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by F. W. Gehring [4].

The Grötzsch ring domain. For $0<r<\infty$, let

$R_G(r)=B^n\setminus\{(x_1,x_2,\cdots,x_n):0\leqslant x_1\leqslant r,\ x_2=x_3=\cdots=x_n=0\}$. The domain $R_G(r)$ is called *the Grötzsch ring domain* corresponding to r. Let $\mu_G(r)$ denote the modulus of the family of arcs joining the boundary components of $R_G(r)$. Then (see [5])

$$\mu_G(r)=\frac{\omega_{n-1}}{[\log\Phi(\frac{1}{r})]^{n-1}} \tag{2.1}$$

where $\omega_{n-1}=m_{n-1}(S^{n-1})$. Using the inequality of F. W. Gehring in [6]

$$a\leqslant\Phi(a)\leqslant\lambda_n a \tag{2.2}$$

we obtain from (2. 1)

$$\frac{\omega_{n-1}}{(\log\frac{\lambda_n}{r})^{n-1}}\leqslant\mu_G(r)\leqslant\frac{\omega_{n-1}}{(\log\frac{1}{r})^{n-1}} \tag{2.3}$$

where $\lambda_n\in[4,2e^{n-1}]$ is the Grötzsch constant.

The Teichmller ring domain. For $r>0$ let

$$R_T(r)=\overline{R}^n\setminus\{(x_1,x_2,\cdots,x_n):-1\leqslant x_1\leqslant 0\ or\ r\leqslant x_1<\infty,\ x_2=x_3=\cdots=x_n=0\}.$$

The domain $R_T(r)$ is called the *Teichmüller ring domain* corresponding to r. Let $\mu_T(r)$ denote the modulus of the family of arcs joining the boundary components of $R_T(r)$. Then (see [5])

$$\mu_T(r)=2^{1-n}\mu_G\left(\sqrt{\frac{1}{1+r}}\right) \tag{2.4}$$

By (2. 3) and (2. 4) we can obtain

$$\frac{\omega_{n-1}}{[\log\lambda_n^2(1+r)]^{n-1}}\leqslant\mu_T(r)\leqslant\frac{\omega_{n-1}}{[\log(1+r)]^{n-1}} \tag{2.5}$$

Next let $D\subset R^n$ is a bounded M−QED domain, E、F are two disjoint continua in closed sets D, the lower bound estimate of mod $[\triangle(E,F;D)]$ play an important role in the proof of theorem 1. Now let us estimate it' s lower bounded.

Since D is bounded, hence both E and F are bounded also, taking $a,b\in E$, $c,d\in F$ such that $dia(E)=|a-b|$, $dia(F)=|c-d|$, *by* [5, 7.35] we have

$$\mathrm{mod}[\triangle(E,F;\overline{R}^n]\geqslant\mu_T\left(\frac{|a-c||b-d|}{|a-b||c-d|}\right) \tag{2.6}$$

Combining (1. 2), (2. 5) and (2. 6) we get

$$\mathrm{mod}[\triangle(E,F;D)]\geqslant\omega_{n-1}/M\log\left[\frac{2\lambda_n^2\mathrm{dia}^2(D)}{\mathrm{dia}(E)\mathrm{dia}(F)}\right]^{n-1} \tag{2.7}$$

3 Proof of theorem 1

Suppose that $f\notin Lip_\alpha(D)$, then there exist sequences $\{x_j\}$ and $\{y_j\}$ in D such that

$$\frac{|f(x_j)-f(y_j)|}{|x_j-y_j|^{\alpha}}\to\infty,\quad as\ j\to\infty.$$

If at least one of the sequence $\{x_j\}$ and $\{y_j\}$ is bounded, without loss of generality, we may assume $\{x_j\}$ is bounded. Then there exist a subsequence of $\{x_j\}$, still denote by $\{x_j\}$ such that $x_j\to x_0\in\overline{D}$ as $j\to\infty$ There are following two cases:

(i) $x_0\in D$, let $d_1=dist(x_0,\partial D)$, taking j sufficient large such that dist $(x_j,\partial D)>\frac{7}{8}d_1$.

If $|x_j-y_j|<\frac{1}{4}d_1$, *let* $0<\varepsilon<\frac{1}{2}|x_j-y_j|$, $R=B^n(x_j,\frac{7}{8}d_1-\varepsilon)\setminus\overline{B}^n(x_j,|x_j-y_j|+\varepsilon)\subset D$, Γ denote the family of arcs joining $B^n(x_j,|x_j-y_j|+\varepsilon)$ and $D\setminus B^n(x_j,\frac{7}{8}d_1-\varepsilon)$ in D. Then by [4, 5.10], (2.7) and the K-quasiconformality of f, we have

$$\frac{1}{KM}\frac{\omega_{n-1}}{\left[\log\dfrac{2\lambda_n^2\mathrm{dia}^2(D')}{\mathrm{dia}(D')|f(x_j)-f(y_j)|}\right]^{n-1}}\leqslant\mathrm{mod}\Gamma=\frac{\omega_{n-1}}{\left[\log\dfrac{\frac{7}{8}d_1-\varepsilon}{|x_j-y_j|+\varepsilon}\right]^{n-1}}\tag{3.1}$$

and hence

$$|f(x_j)-f(y_j)|\leqslant\frac{2\lambda_n^2\mathrm{dia}(D')}{\left[\frac{d_1}{2}\right]^{(KM)^{1/1-n}}}|x_j-y_j|^{(KM)^{1/1-n}}$$

$$\leqslant\frac{2\lambda_n^2\mathrm{dia}(D')}{\left[\frac{d_1}{2}\right]^{(KM)^{1/1-n}}}\left[\frac{d_1}{4}\right]^{(KM)^{1/1-n}-\alpha}|x_j-y_j|^{\alpha}\tag{3.2}$$

If $|x_j-y_j|\geqslant\frac{1}{4}d_1$, then by the boundedness of D', we have

$$|f(x_j)-f(y_j)|\leqslant\frac{2\lambda_n^2\mathrm{dia}(D')}{\left(\frac{1}{2}d_1\right)^{\alpha}}|x_j-y_j|^{\alpha}\tag{3.3}$$

(ii) $x_0\in\partial D$, fixed a continunm $A\subset D$, let $d_2=dist(A,\partial D)$, taking j sufficient large such that dist $(x_j,A)>\frac{7}{8}d_2$.

If $|x_j-y_j|<\frac{1}{8}d_2$, *let* $0<\varepsilon<\frac{1}{2}|x_j-y_j|$, $R=B^n(x_j,|x_j-y_j|+\varepsilon)\setminus\overline{B}^n(x_j,|x_j-y_j|+\frac{1}{2}\varepsilon)$, then R separates points x_j,y_j and A in $\overline{R}^n$, let γ_j be the open segment joining x_j and y_j. Then there are two cases:

(a) $\gamma_j\subset\overline{D}$. let A_j be the component of $D\setminus R$ which contain x_j and y_j, then by (2.7), the K-quasiconformality of f and the compare principle of modulus we get

$$\frac{1}{KM}\frac{\omega_{n-1}}{\left[\log\dfrac{2\lambda_n^2\mathrm{dia}^2(D')}{\mathrm{dia}f(A)|f(x_j)-f(y_j)|}\right]^{n-1}}\leqslant\mathrm{mod}[\triangle(A,A_j;D)]$$

$$\leqslant \frac{\omega_{n-1}}{\left[\log \frac{\frac{7}{8}d_2 - \varepsilon}{|x_j - y_j| + \varepsilon}\right]^{n-1}} \leqslant \frac{\omega_{n-1}}{\left[\log \frac{\frac{1}{2}d_2}{|x_j - y_j|}\right]^{n-1}} \tag{3.4}$$

and hence

$$|f(x_j) - f(y_j)| \leqslant \frac{2\lambda_n^2 \mathrm{dia}^2(D')}{\left[\frac{1}{2}d_2\right]^{(KM)^{1/1-n}} \mathrm{dia}(f(A))} \left[\frac{1}{8}d_2\right]^{(KM)^{1/1-n-\alpha}} |x_j - y_j|^{\alpha} \tag{3.5}$$

If $|x_j - y_j| \geqslant \frac{1}{8}d_2$, then by the boundedness of D' we have

$$|f(x_j) - (y_j)| \leqslant \frac{\mathrm{dia}(D')}{(\frac{1}{8}d_2)^{\alpha}} |x_j - y_j|^{\alpha} \tag{3.6}$$

(b) $\gamma_j \not\subset \overline{D}$, then $\gamma_j \cap \partial D \neq \varnothing$ and there exist at least two points of $\gamma_j \cap \partial D$. Let x_j' and y_j' be the nearest points from x_j and y_j respectively, c_j be the open segment joining x_j to x_j', d_j be the open segment joining y_j to y_j', then $c_j \subset \overline{D}$ and $d_j \subset \overline{D}$, the detail proof similar to (a), we can prove that there exist constant M_1 and M_2 satisfying

$$|f(x_j) - f(x_j')| \leqslant M_1 |x_j - x_j'|^{\alpha} \leqslant M_1 |x_j - y_j|^{\alpha} \tag{3.7}$$

$$|f(y_j) - f(y_j')| \leqslant M_2 |y_j - y_j'|^{\alpha} \leqslant M_2 |x_j - y_j|^{\alpha} \tag{3.8}$$

since $f \in Lip_{\alpha}(\partial D)$, hence there exists constant $M_3 > 0$ satisfying

$$|f(x_j') - f(y_j')| \leqslant M_3 |x_j' - y_j'|^{\alpha} \leqslant M_3 |x_j - y_j|^{\alpha} \tag{3.9}$$

Combining (3.7)、(3.8)、(3.9) and by the triangle inequality we have

$$|f(x_j) - f(y_j)| \leqslant (M_1 + M_2 + M_3) |x_j - y_j|^{\alpha} \tag{3.10}$$

If sequences $\{x_j\}$ and $\{y_j\}$ are both unbounded, then then exist subsequence $\{x_j'\} \subset \{x_j\}$ and $\{y_j'\} \subset \{y_j\}$ such that $x_j' \to \infty$ and $y_j' \to \infty$ as $j \to \infty$. For convinence, we still instead $\{x_j\}$ for $\{x_j'\}$ and $\{y_j\}$ for $\{y_j'\}$.

Fixed a continuum $A \subset D$, let $R > 0$ be sufficient large, such that $A \subset B^n(R)$, let j be sufficient large such that $x_j, y_j \notin B^n(2R)$.

First we assume $|x_j - y_j| < \frac{1}{4}R$, taking $0 < \varepsilon < \frac{1}{2}|x_j - y_j|$, $R_j = B^n(x_j, |x_j - y_j| + \varepsilon) \backslash \overline{B}^n(x_j, |x_j - y_j| + \frac{1}{2}\varepsilon)$, let γ_j be the open segment joining x_j to y_j, then there are two cases:

(a′) $\gamma_j \subset \overline{D}$. Let A_j be a component of $D \backslash R_j$ which contain x_j and y_j, Γ_j denote the family of arcs joining A and A_j in D, then we have

$$\frac{1}{M} \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2 \mathrm{dia}^2(D')}{\mathrm{dia}(f(A))|f(x_j) - f(y_j)|}\right)^{n-1}} \leqslant K \mathrm{mod} \Gamma \leqslant \frac{K\omega_{n-1}}{\left(\log \frac{2R}{3|x_j - y_j|}\right)^{n-1}} \tag{3.11}$$

and hence

$$|f(x_j)-f(y_j)|\leqslant\frac{2\lambda_n^2\mathrm{dia}^2(D')}{\left[\frac{2}{3}R\right]^{(KM)^{1/1-n}}\mathrm{dia}(f(A))}\left[\frac{1}{4}R\right]^{(KM)^{1/1-n}-\alpha}|x_j-y_j|^\alpha$$

(b′) $\gamma_j\not\subset\overline{D}$. Then $\gamma_j\cap\partial D\neq\varnothing$ and there exist at least two points of $\gamma_j\cap\partial D$. Let x_j' and y_j' be the nearest points from x_j and y_j respectively. Denote by ξ_j and η_j the open segments joining x_j to x_j' and y_j to y_j' respectively, then $\xi_j\subset\overline{D}$ and conclude that there exists constant $M'>0$ satisfying

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If $|x_j-y_j|\geqslant\frac{R}{4}$, then by the boundness of D' we have

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According to the above discuss, we conclude that there exists constand $M>0$ satisfying $|f(x_j)-f(y_j)|/|x_j-y_j|^\alpha\leqslant M$ for sufficient large j, this contradicts with $|f(x_j)-f(y_j)|/|x_j-y_j|^\alpha\to\infty$. Thus $f\in Lip_\alpha(D)$.

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References

1 Gehring F W. Martio O. Lipschitz classes and quasiconformal mappings. Ann Acad Sci Fenn Ser Math, 1988, 10: 203～219

2 Gehring F W, Hayman W K, Hinkkanen A. Analytic functions satisfying Hölder conditions on the boundary. J Approx Theory, 1982, 35: 353～361

3 Väisälä J. Lectures on n－dimensional quasiconformal mappings. Lecture Notes in Math, Springer－Verlag, Berlin and New York, 1971, 229

4 Gehring F. W. Rings and quasiconformal mappings in space. Trans Amer Math Soc, 1962, 103

5 Vuorinen M. Conformal geometry and quasiregular mappings. Lecture Notes in Math. Springer－Verlag, Berlin and New york, 1988, 1319

6 Gehring F W. Symmetrization of rings in space. Trans Amer Math. Soc 1961, 101: 499～519

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摘　要　设 f 是 R^n 中的域 D 到有界的 $M-QED$ 域上的 $K-$拟共形映照，$0<\alpha\leqslant(KM)^{1/1-n}$. 在本文中作者证明了 $f\in Lpi_\alpha(\partial D)$ 的充要条件是 $f\in Lpi_\alpha(D)$。

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