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The Generalized Problem of Hanoi Tower And Its Optimal Solving Algorithm*

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Abstract Through researching into the generalized problem of Hanoi tower, this paper has revealed the optimal removing sequence of the circle plates during its solving procedure. And through analyzing the algorithm complexity, this paper has put forward an optimal solving algorithm and drawn into an optimal general solution.

Key words Hanoi tower, optimal solving algorithm, algorithm complexity

1 The Classical Problem of Hanoi Tower and Its Solving Algorithm

1.1 The Classical Problem of Hanoi Tower

Suppose that there are three tower pillars, A, B and C, which are called source pillar, auxiliary pillar and target pillar, respectively; there are n different circle plates which are numbered from small to large as 1, 2, …, n, respectively.

Initial state: the n circle plates are at pillar A;

Target state: the n circle plates are at pillar C.

Realizig the removing procedure from the initial state to the target state according to the following three rules:

(1) Only one plate can be removed at any time.

(2) One plate can be removed from one pillar to another.

(3) A larger plate can never be put upon another smaller one.

1.2 The Solving Algorithm

If $n=1$, then the unique plate $n(=1)$ can be removed directly from A to C; otherwise, the removing procedure can be implemented through the following three steps:

(1) Helping with C, remove the upper $(n-1)$ plates from A to B.

(2) Remove directly the bottom plate n from A to C.

(3) Helping with A, remove the $(n-1)$ plates from B to C.

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1.3 The Analyzing of Alogorithm Complexity

Note $M_1(n)$ as the removing times of plates during the solving procedure of the n-order classical problem of Hanoi tower. According to the solving algorithm above, we have:

$M_1(1)=1$; $M_1(n)=2M_1(n-1)+1$.

With the method of mathematical induction, easy to prove that:

$M_1(n)=2^n-1, (n\geq 1)$.

2 The 2-auxiliary-pillar Problem of Hanoi Tower and Its Solving Algorithm

2.1 The 2-auxiliary-pillar Problem of Hanoi Tower

Compare with the classical problem of Hanoi tower, which has only one auxiliary pillar, the 2-auxiliary-pillar problem of Hanoi tower has two auxiliary pillars B_1, B_2 and keeps other characters.

2.2 The Solving Algorithm

If $n=1$; then the unique plate $n(=1)$ can be removed directly from A to C; otherwise, the removing procedure can be implemented through the following five steps:

(1)Helping with C, B_2, remove the plates $1\sim k_1$ from A to B_1.

(2)Helping with C, remove the plates $(k_1+1)\sim(n-1)$ from A to B_2.

(3)Remove directly the bottom plate n from A to C.

(4)Helping with A, remove the plates $(k_1+1)\sim(n-1)$ from B_2 to C.

(5)Helping with A, B_2, remove the plates $1\sim k_1$ from B_1 to C.

2.3 The Analzing of Algorithm Complexity

Note $M_2(n)$ as the removing times of plates during the solving procedure of the n-order 2-auxiliary-pillar problem of Hanoi tower. According to the solving algorithm above, we have:

$M_2(1)=1$; $M_2(n)=2M_2(k_1)+2M_1(k_2)+1$; $(n=k_1+k_2+1;\ k_1,\ k_2\geq 0)$.

Expand n as the following form:

$n=1+2+\cdots+(p-1)+p+r, (p\geq 0;\ r\in\{1,\ 2,\ \cdots,\ p+1\})$

(1)$r=p+1$: $n=1+2+\cdots+(p-1)+p+(p+1)$.

Let $k_2=p$, then $k_1=n-k_2-1=1+2+\cdots+(p-1)+p$;

To k_1, let the new $k_2=p-1$, then the new $k_1=1+2+\cdots+(p-2)+(p-1)$.

According to the method of getting k_1, k_2 above, continue the procedure until $k_2=1$, $k_1=1$. Therefore, we have the following inference procedure:

$M_2(n)$

$=2M_2(1+2+\cdots(p-1)+p)+2M_1(p)+1$

$=\cdots$

$=2^pM_2(1)+2^pM_1(1)+2^{(p-1)}M_1(2)+\cdots+2^1M_1(p)+1+2^1+\cdots+2^{(p-1)}$

$=p2^{(p+1)}+1.$

(2) $r\in\{1,2,,p\}$: $n=1+2+\cdots+(p-1)+p+r.$

Let $k_2=p$, then $k_1=n-k_2-1=1+2+\cdots+(p-1)+(r-1)$;

To k_1, let the new $k_2=p-1$, then the new $k_1=1+2+\cdots+(p-2)+(r-2)$.

According to the method of getting k_1, k_2 above, continue the procedure until $k_2=p-(r-1)$, $k_1=1+2+\cdots+(p-r-1)+(p-r)$. From now on, we have the similar case in (1), where k_2 will be successively $(p-r-1)$, $\cdots$, 2, 1. Therefore, we have the following inference procedure:

$M_2(n)$

$=2M_2(1+2+\cdots(p-1)+(r-1))+2M_1(p)+1$

$=\cdots$

$=2^rM_2(1+2+\cdots+(p-r-1)+(p-r))+2^rM_1(p-r+1)+\cdots+2^1M_1(p)$

$+1+2^1+\cdots+2^{(r-1)}$

$=2^p(p+r-1)+1.$

From (1), (2), we have the following conclusion:

$M_2(n)=2^p(p+r-1)+1$;

$n=1+2+\cdots+(p-1)+p+r$; $(p\geq 0;\ r\in\{1,\ 2,\ \cdots,\ p+1\})$.

Here, $p\in[\sqrt{2n}-2,\ \sqrt{2(n-1)}+1]$; $(n\geq 2)$.

With the method of mathematical induction, we can prove the conclusion above.

2.4 The Proof of the Optimal Solving Algorithm

Note $M_2^*(n)=2^p(p+r-1)+1$; $n=1+2+\cdots+(p-1)+p+r$, $(p\geq 0; r\in\{1,2,\cdots,p+1\})$. During the solving procedure of $M_2(n)$ with a recurrent method, as long as there is one step at which k_1, k_2 are different with 2.3(1) or (2), there will be the result: $M_2(n)\geq M_2^*(n)$. The algorithm is called the optimal algorithm to solve the 2-auxiliary-pillar problem of Hanoi tower, by which k_1, k_2 are selected according to 2.3(1)(2).

The removing of k_1 plates helping with two auxiliary pillars, k_2 plates helping with one auxiliary pillar; so the optimal algorithm demand $k_1\geq k_2\geq 0$ at least.

(1) $k_2\in\{0,\cdots,p-1\}$: $p\geq 1$.

Case 1: $k_2=p-1$; $k_1=n-k_2-1=1+2+\cdots+(p-1)+r$, $r\in\{1,2,\cdots,p+1\}$.

(i) $r\in\{1,2,\cdots,p\}$:

$M_2(n)=2[2^{(p-1)}((p-1)+r-1)+1]+2[2^{(p-1)}-1]+1$

$=2^p(p+r-1)+1=M_2^*(n)$

(ii) $r=p+1$: $k_1=n-k_2-1=1+2+\cdots+(p-1)+p+1$;

$M_2(n)=2[2^p(p+1-1)+1]+2[2^{(p-1)}-1]+1$

$=2^p[p+(p+1)-1)]+1+2^p$

$=M_2^*(n)+2^p>M_2^*(n).$

Case 2: $k_2\in\{0,\ \cdots,\ p-2\}$: $p\geq 2$.

Let $k_2=p-t$, $t\in\{2, \cdots, p\}$;

then $k_1=n-k_2-1=1+2+\cdots+(p-1)+(r+t-1)$; $r+t-1\in\{2, \cdots, 2p\}$.

(i) $r+t-1\in\{2, \cdots, p\}$:

$$M_2(n)=2[2^{(p-1)}((p-1)+(r+t-1)-1+1]+2[2^{(p-1)}-1]+1$$
$$=2^p(p+r+t-3)+2^{(p-t+1)}+1$$
$$=2^p(p+r-1)+1+2^p(t-2)+2^{(p-t+1)}>M_2^*(n).$$

(ii) $r+t-1\in\{(p+1), \cdots, 2p\}$: $k_1=n-k_2-1=1+2+\cdots+(p-1)+p+(r+t-p-1)$;

$$M_2(n)=2[2^p(p+(r-t-p-1)-1)+1]+2[2^{(p-t)}-1]+1$$
$$=2^p(2r+2t-4)+2^{(p-t+1)}+1$$
$$=2^p(p+r-1)+1+2^p(r+2t-p-3)+2^{(p-t+1)}>M_2^*(n).$$

From above, we have $M_2(n)>M_2^*(n)$ when $k_2\leq p-1$.

(2) $k_2\in\{p+1, \cdots, (n-1)/2\}$: $p\geq 0$.

Case 1: $k_2=p+1$; $k_1=1+2+\cdots+(p-1)+(r-2)$; $r\in\{1, 2, \cdots, p+1\}$.

(i) $r\in\{1, 2\}$: $k_1=1+2+\cdots+(p-2)+(p+r-3)$; $(p+r-3)\in\{p-2, p-1\}$.

$$M_2(n)=2[2^{(p-2)}((p-2)+(p+r-3)-1)+1]+2[2^{(p+1)}-1]+1$$
$$=2^{(p-1)}(2p+r-6)+2^{(p+2)}+1$$
$$=2^p(p+r-1)+1+2^{(p-1)}(4-r)>M_2^*(n)$$

(ii) $r\in\{3, \cdots, p+1\}$: $(r-2)\in\{1, \cdots, p-1\}$.

$$M_2(n)=2[2^p(p-1)+(r-2)-1)+1]+2[2^{(p+1)}-1]+1$$
$$=2^p(p+r-1)+1+2^p(p+r-3)>M_2^*(n).$$

Case 2: $k_2\in\{p+1, \cdots, (n-1)/2\}$.

Let $k_2=p+t$, $t\in\{1, \cdots, (n-1)/2-p\}$. Suppose that $k_1=n-k_2-1=1+2+\cdots+(q-1)+q+s>q$; $q\in\{0, \cdots, p-1\}$; $s\in\{1, \cdots, q+1\}$.

Let $k_2'=k_2+1$, $k_1'=k_1-1\geq 0$;

then
$$M_2'(n)-M_2(n)=2[2M_2'(k_1')+2M_1(k_2')+1]-[2M_2(k_1)+2M_1(k_2)+1]$$
$$=2[M_2(k_1-1)-M_2(k_1)]+2[M_1(k_2+1)-M_1(k_2)]$$
$$=2^{k_2+1}-2[M_2(k_1)-M_2(k_1-1)]$$
$$=2^{k_2+1}-2[(2^q(q+s-1)+1)-2^q(q+(s-1)-1)+1)]$$
$$=2^{k_2+1}-2^{q+1}>0.$$

$M_2(n)$ will be increased monotonously when $k_2\geq p+1$, k_2 increases 1 and k_1 decreases 1; we have $M_2(n)>M_2^*(n)$ when $k_2\geq p+1$.

From (1)(2) above, we have the following result:

$M_2(n)$ has the smallest value $M_2^*(n)=2^p(p+r-1)+1$ when $k_2=p$; that is, $k_2=p$, $k_1=n-p-1$ is the optimal solution of the 2-auxiliary-pillar problem of Hanoi tower. Here, $n=1+2+\cdots+(p-1)+p+r$; ($p\geq 0$; $r\in\{1, 2, \cdots, p+1\}$).

3 The Generalized Problem of Hanoi Tower and Its Optimal Solving Algorithm

3.1 The Generalized Problem of Hanoi Tower

Compare with the classical problem of Hanoi tower, which has only one auxiliary pillar, the generalized problem of Hanoi tower has two or more auxiliary pillars B_1, …, $B_m (m \geq 2)$ and keeps other characters.

3.2 The Solving Algorithm

If $n=1$, then the unique plate $n(=1)$ can be removed directly from A to C; otherwise, the removing procedure can be implemented through the following $(2m+1)$ steps:

(1)Helping with C, B_2, …, B_m, remove the plates $1 \sim k_1$ from A to B_1.

(2)Helping with C, B_3, …, B_m, remove the plates $(k_1+1) \sim (k_1+k_2)$ from A to B_2.

⋮

(m)Helping with C, remove the plates $(k_1+\cdots+k_{m-1}+1) \sim (n-1)$ from A to B_m.

(m+1)Remove directly the bottom plate n from A to C.

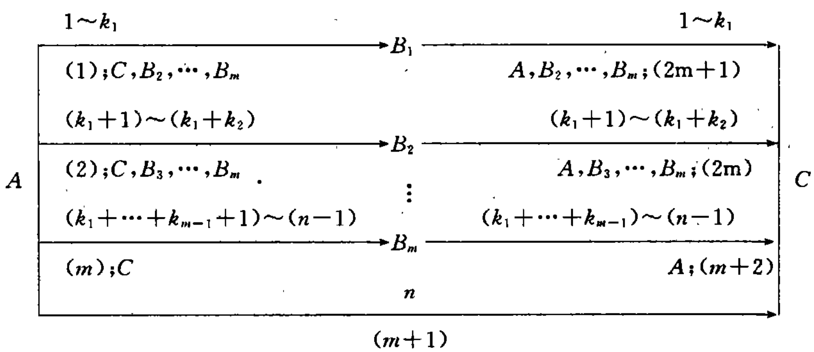
(m+2)Helping with A, remove $(k_1+\cdots+k_{m-1}+1) \sim (n-1)$ from B_m to C.

⋮

(2m)Helping with A, B_3, …, B_m, remove the plates $(k_1+1) \sim (k_1+k_2)$ from B_2 to C.

(2m+1)Helping with A, B_2, …, B_m, remove the plates $1 \sim k_1$ from B_1 to C.

The procedure above can be described as the following picture:



3.3 The Optimal Solving Algorithm and Its Analyzing of Complexity

Note $M_m(n)$ as the removing times of plates during the solving procedure of the n-order generalized problem of Hanoi tower. According to the solving algorithm above, we have:

$M_m(1)=1$;

$M_m(n)=2M_m(k_1)+2M_{m-1}(k_2)+\cdots+2M_2(k_{m-1})+2M_1(k_m)+1$;

$n=k_1+k_2+\cdots+k_m+1$; $k_1, k_2, \cdots, k_m \geq 0$.

Since the inference procedure will become much more complex with an increasing value of m, here we will give out only the corresponding conclusions of the n-order generalized problem of Hanoi tower.

Through developing n as a form of a front part of $(m-1)$-order natural series and a remainder, we can get the general solution of the n-order generalized problem of Hanoi tower as followings:

$M_m(n)=2^p[F(m,p)+r+(-1)^{m-1}]+(-1)^m$; $m \geq 2$.

$F(1,p)=0$;

$F(2,p)=p$;

$F(3,p)=(p-1)p/2$;

$$F(m,p)=p(p+1)\cdots(p+m-2)/(m-1)!-F(m-1,p)$$
$$=(p-1)p(p+1)\cdots(p+m-3)/(m-1)!+F(m-2,p).$$

Here, $p \in [\sqrt[m]{m!\,n}-(m-1), \sqrt[m]{m!\,(n-1)}-(m-3)]$.

4 Conclusion

This paper has revealed the optimal removing sequence of the circle plates during the solving procedure; and put forward an optimal solving algorithm to the generalized problem of Hanoi tower.

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广义 Hanoi 塔问题及其最佳求解算法研究

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摘　要　本文通过研究广义 Hanoi 塔问题，揭示了其求解过程中圆盘移动的最佳次序，提出了相应的最佳求解算法；并通过分析其算法的复杂度，给出了 n 阶广义 Hanoi 塔问题的最佳通解。

关键词　最佳求解算法，算法复杂度

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