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薄壳稳定的变分原理*

黄　炎

(国防科技大学航天技术系　长沙　410073)

摘　要　按照易曲物体的形变理论来确定薄壳的内力和内矩、变形位能以及外力的功，根据虚位移原理求得临界载荷的能量准则，并导出稳定问题的平衡方程和边界条件。对公式进行了合理的分析和简化。

关键词　薄壳，稳定，临界载荷

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The Variational Principle of Stability of Thin Shells

Huang Yan

(Department of Astronantic Technology, NUDT, Changsha, 410073)

Abstract This paper gives the resultant forces and moments, strain energy and the work of the external forces on the basis of the deformation theory of flexible body. In accordance with the principle of virtual displacement, the energy criterion of critical load is obtained and the equilibrium diffenential equations and boundary conditions of stability problem are derived. The formula is discussed and simplified.

Key words thin shell, stability, critical load

用能量法求解薄壳的稳定问题，一般均采用扁壳理论[1]，不能精确地解决曲率大的薄壳稳定问题。用精确的平衡方程的方法尚难求解比较复杂的稳定问题[2]。应用变分学的直接方法可以有效地解决薄壳稳定的具体问题。

1　薄壳的变形和变形位能

按照易曲物体的形变理论[3]，薄壳任一点在坐标线α_1、α_2和中曲面法线方向的位移$\hat{u}$、$\hat{v}$、$\hat{w}$可表为

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$$\hat{u} = u + z\theta, \hat{v} = v + z\varphi, \hat{w} = w + z\chi \tag{1}$$

式中u、v、w 为壳体中曲面的位移。由直法线假设可得[4]

$$\left.\begin{aligned} \theta &= -e_{13}(1-e_{11}) + e_{12}e_{23} \\ \psi &= -e_{23}(1-e_{22}) + e_{21}e_{13} \\ \chi &= -\frac{1}{2}(e_{13}^2 + e_{23}^2) \end{aligned}\right\} \tag{2}$$

式中

$$e_{11} = \frac{1}{A_1}\frac{u}{\alpha_1} + \frac{1}{A_1A_2}\frac{A_1}{\alpha_2}v + \frac{W}{R_1}, \cdots \tag{3}$$

壳体任一点的伸长度$\hat{\epsilon}_{11}$、$\hat{\epsilon}_{22}$和切应变$\hat{\epsilon}_{12}$分别为

$$\hat{\epsilon} = \epsilon_{11} + z\chi_{11}, \hat{\epsilon}_{22} = \epsilon_{22} + z\chi_{22}, \hat{\epsilon}_{12} = \epsilon_{12} + z\chi_{12} \tag{4}$$

式中

$$\epsilon_{11} = e_{11} + \frac{1}{2}(e_{11}^2 + e_{12}^2 + e_{13}^2), \cdots \tag{5}$$

$$\chi_{11} = k_{11} + e_{11}k_{11} + e_{12}k_{12} + e_{13}k_{13}, \cdots \tag{6}$$

$$k_{11} = \frac{1}{A_1}\frac{\theta}{\alpha_1} + \frac{1}{A_1A_2}\frac{A_1}{\alpha_2}\psi + \frac{\chi}{R_1}, \cdots \tag{7}$$

当忽略壳体中曲面法向应力σ_{33}时，壳体的内力T_1、T_2、T_{12}和内矩M_{11}、M_2、M_{12}以及变形位能A 为[5]

$$T_1 = \frac{E\delta}{1-\mu^2}(\epsilon_{11} + \mu\epsilon_{22}), \cdots \tag{8}$$

$$\begin{aligned} A = \frac{E\delta}{z(1-\mu^2)}\iint\Big[&\epsilon_{11}^2 + \epsilon_{22}^2 + 2\mu\epsilon_{11}\epsilon_{22} + \frac{1-\mu^2}{2}\epsilon_{12}^2 + \frac{\delta^2}{12}(\chi_{11}^2 + \chi_{22}^2 \\ &+ 2\mu\chi_{11}\chi_{22} + \frac{1-\mu}{2}\chi_{12}^2)\Big]A_1A_2\,d\alpha_1 d\alpha_2 \end{aligned} \tag{9}$$

应用到（8）式，上式可写为

$$A = \frac{1}{2}\iint(T_1\epsilon_{11} + T_2\epsilon_{22} + T_{12}\epsilon_{12} + M_1\chi_{11} + M_2\chi_{22} + M_{12}\chi_{12})A_1A_2\,d\alpha_1 d\alpha_2 \tag{10}$$

将虚位移原理应用到处于平衡状态的已变形物体上，使位移u、v、w 得到一虚位移δu、δv、δw。此时变形位能得到一增量δA 为[5]

$$\begin{aligned} \delta A = \frac{1}{2}\iint(&T_1\delta\epsilon_{11} + T_2\delta\epsilon_{22} + T_{12}\delta\epsilon_{12} + M_1\delta\chi_{11} + M_2\delta\chi_{22} + \\ &M_{12}\delta\chi_{12})A_1A_2\,d\alpha_1 d\alpha_2 \end{aligned} \tag{11}$$

增量δA 应等于作用于壳体上的全部外力在指定的虚位移上所作的功

$$\delta A = \delta R_1 + \delta R_2 \tag{12}$$

式中

$$\delta R_1 = \iint(q_1^*\delta v + q_2^*\delta u + q_n^*\delta w)A_1A_2(1+e_{11})(1+e_{22})\,d\alpha_1 d\alpha_2 \tag{13}$$

$$q_1^* = q_1 + q_1 e_{11} + q_2 e_{21} - q_n e_{13}, \cdots \tag{14}$$

$$\delta R_2 = \int (\overline{T}_1 \mathrm{d}u + \overline{T}_{12}\delta v + \overline{N}_1 \delta w + \overline{M}_1 \delta\theta) A_2 \mathrm{d}\alpha_1 + \int (\overline{T}_{21}\delta u + \overline{T}_2 \delta v + \overline{N}_2 \delta w + \overline{M}_2 \delta\psi) A_1 \mathrm{d}\alpha_1 + \Sigma \overline{P} \delta w \tag{15}$$

2 确定临界载荷的能量准则

设壳体的载荷等于临界载荷，则壳体将同时有两个可能的无限接近的平衡位置。设初始平衡位置处于薄膜应力状态，此时壳体中曲面各点的位移为 u_0、v_0、w_0，而另一个新的平衡位置的位移为

$$u = u_0 + \alpha u_1, v = v_0 + \alpha v_1, w = w_0 + \alpha w_1 \tag{16}$$

式中 αu_1、αv_1、αw_1 为中曲面各点从初始平衡位置移到新的平衡位置时的附加位移，而且假定 u_1、v_1、w_1 是有限的。α 是一个不依赖于 α_1 和 α_2 的无限小量。将 (16) 式代入 (3) 式可得

$$e_{11} = e_{11}^0 + \alpha e_{11}^{\prime}, \cdots \tag{17}$$

e_{11}^0 和 $e_{11}^{\prime}$ 的表示式为将 e_{11} 表示式中的 u、v、w 分别改为 u_0、v_0、w_0 和 u_1、v_1、w_1。将上式代入 (2) 式和 (4) 式可得

$$\theta = \theta^0 + \alpha\theta^{\prime} + \alpha^2\theta^{\prime\prime}, \cdots \tag{18}$$

$$\epsilon_{11} = \epsilon_{11}^0 + \alpha\epsilon_{11}^{\prime} + \alpha^2\epsilon_{11}^{\prime\prime}, \cdots \tag{19}$$

不难写出 θ^0 等的表示式，但由于经典理论位置可用于初始平衡位置，即 e_{11}^0 等和 1 相比是可以略去的，故简化后可得

$$\theta^0 = - e_{13}^0, \theta^{\prime} = - e_{13}^{\prime}, \cdots \tag{18′}$$

$$\epsilon_{11}^0 = e_{11}^0, \epsilon_{11}^{\prime} = e_{11}^{\prime}, \cdots \tag{19′}$$

$$k_{11} = k_{11}^0 + \alpha k_{11}^{\prime} + \alpha^2 k_{11}^{\prime\prime}, \cdots \tag{20}$$

将 (17) 式和 (20) 式代入 (6) 式，简化后并略去 α^2 的项可得

$$\chi_{11} = \chi_{11}^0 + \alpha\chi_{11}^{\prime} + \alpha\chi_{11}^{\prime\prime}, \cdots \tag{21}$$

$$\chi_{11}^0 = k_{11}^0 = - \frac{1}{A_1}\frac{\partial e_{13}^0}{\partial\alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2} e_{23}^0, \cdots$$

$$\chi_{11}^{\prime} = k_{11}^{\prime} = - \frac{1}{A}\frac{\partial e_{13}^{\prime}}{\partial\alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2} e_{23}^{\prime}, \cdots$$

将 (19) 式和 (20) 式代入 (8) 式，并注意到初始平衡位置为薄膜应力状态，即 $M_1^0 = M_2^0 = M_{12}^0 = 0$，相应地 $\chi_{11}^0 = \chi_{22}^0 = \chi_{12}^0 = 0$ 可得

$$T_1 = T_1^0 + \alpha T_1^{\prime} + \alpha^2 T_1^{\prime\prime}, \cdots \tag{22}$$

$$M_1 = \alpha M_1^{\prime} + \alpha^2 M_1^{\prime\prime}, \cdots \tag{23}$$

将 (19) 式，(21) 式，(22) 式和 (23) 式代入 (10) 式，略去 α^3 及以上的项可得

$$A = A^0 + \alpha A^{\prime} + \alpha A^{\prime\prime}$$

$$A^{\prime} = \iint [T_1^0 e_{11}^{\prime} + T_2^0 e_{22}^{\prime} + T_{12}^0 (e_{12}^{\prime} + e_{21}^{\prime})] A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{24}$$

$$A'' = \iint\{\frac{1}{2}[T_1'e_{12}' + T_2'e_{22}' + T_{12}'(e_{12}' + e_{21}') + M_1'k_{11}' + M_2^1k_{22}^1 + M_{12}'$$
$$(k_{12}' + k_{21}') + T_1^0(e_{12}' + e_{11}' + e_{13}') + T_2^0(e_{21}' + e_{22}' + e_{23}')] + T_{12}^0(e_{11}'e_{21}'$$
$$+ e_{12}'e_{22}' + e_{13}'e_{23}')\}A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{25}$$

设应用虚位移原理时，只变化附加位移 αu_1、αv_1、αw_1，而不变化 u_0、v_0、w_0，则

$$\delta u = \alpha\delta u_1, \delta v = \alpha\delta v_1, \delta w = \alpha\delta w_1 \tag{26}$$

通常稳定问题中壳体上的载荷是指与壳体变形有关的表面力。将 (17) 式代入 (14) 式，然后代入 (13) 式，略去 α^3 的项，简化后可得

$$\delta R_1 = \alpha\delta R_1' + \alpha^2\delta R_1''$$
$$\delta R_1' = \iint(q_1\delta u_1 + q_2\delta v_1 + q_n\delta w_1)A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{27}$$
$$\delta R_1'' = \iint\{[q_1(e_{11}' + e_{22}') + q_1e_{11}' + q_2e_{21}' + q_ne_{13}']\delta u_1 + [q_2(e_{11}' + e_{22}')$$
$$+ q_1e_{12}' + q_2e_{22}' - q_ne_{23}']\delta v_1 + [q_n(e_{11}' + e_{22}') + q_1e_{13}' + q_2e_{23}']$$
$$\delta w_1\}A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{28}$$

在给定边界上的载荷，力的大小和方向与壳体的变形无关，由于初始平衡位置为薄膜应力状态，相应地 $\overline{N}_1 = \overline{N}_2 = \overline{M}_1 = \overline{M}_2 = \overline{R} = 0$。将 (26) 式代入 (15) 式可得

$$\delta R_2 = \alpha\delta R_2' = \alpha[\int(\overline{T}_1\delta u_1 + \overline{T}_{12}\delta v_1)A_2\mathrm{d}\alpha_2 + \int(\overline{T}_{21}\delta u_1 + T_2\delta v_1)A_1\mathrm{d}\alpha_1] \tag{29}$$

因此，对新的平衡位置，虚位移原理可写为

$$\delta A = \alpha\delta A' + \alpha^2\delta A'' = \alpha(\delta R' + \delta R_2') + \alpha^2\delta R_1''$$

由于 A^0 不依赖于 u_1、v_1、w_1，故 $\delta A^0 = 0$，而 $\delta A'$、$\delta R_1'$ 和 $\delta R_2'$ 恰好是初始位置的虚位移原理的数学公式，即

$$\delta A' = \delta R_1' + \delta R_2' \tag{30}$$

比较以上二式，可得弹性稳定问题中的变分公式为

$$\delta A'' = \delta R_1'' \tag{31}$$

3 确定临界载荷的微分方程

将 (17) 式代入 (24) 式，然后和 (27) 式，(29) 式一齐代入 (30) 式，可得

$$\delta(A' - R_1' - R_2') = -\iint[(\frac{\partial A_2T_1^0}{\partial\alpha_1} + \frac{\partial A_1T_{12}^0}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}T_{12}^0 - \frac{\partial A_2}{\partial\alpha_1}T_2^0 + A_1A_2q_1)$$
$$\delta u_1 + (\frac{\partial A_2T_{12}^0}{\partial\alpha_1} + \frac{\partial A_1T_2^0}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1}T_{12}^0 - \frac{\partial A_1}{\partial\alpha_2}T_1^0 + A_1A_2q_2)\delta v_1 + A_1A_2(q_n$$
$$-\frac{T_1^0}{R_1} - \frac{T_2^0}{R_2})\delta w_1]\mathrm{d}\alpha_1\alpha_2 + \int[(T_1^0 - \overline{T}_1)\delta u_1 + (T_{12}^0 - \overline{T}_{12})\delta v_1]A_2\mathrm{d}\alpha_2 + \int[$$
$$(T_{12}^0 - \overline{T}_{21})\delta u_1 + (T_2^0 - \overline{T}_2)\delta v_1]A_1\mathrm{d}\alpha_1 = 0 \tag{32}$$

将 (17) 式代入 (25) 式和 (28) 式，然后代入 (31) 式，并进行分部积分，经整理后可得与 (32) 式相当的等式，其中右边双重积分号下各项等于零组成临界载荷的新

的平衡位置的微分方程，它与文献[2]根据壳体平衡条件导得的微分方程是相同的。线积分号下各项等于零以及常数项等于零为力在边界及角点所应满足的条件，即在 α_1 边界上应有

$$T_1^{⑦} + T_1^0 e_{11}^{⑦} + T_{12}^0 e_{21}^{⑦} = 0, \quad T_{12}^{⑦} + \frac{M_{12}^{⑦}}{R_2} + T_1^0 e_{12}^{⑦} + T_{12}^0 e_{22}^{⑦} = 0$$

$$\frac{1}{A_1 A_2}\left(\frac{\partial A_2 M_1^{⑦}}{\partial \alpha_1} + \frac{\partial A_1 M_{12}^{⑦}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} M_{12}^{⑦} - \frac{\partial A_2}{\partial \alpha_1} M_2^{⑦}\right) + \frac{1}{A_2}\frac{\partial M_{12}^{⑦}}{\partial \alpha_2} + T_1^0 e_{13}^{⑦} + T_{12}^0 e_{23}^{⑦} = 0, \quad M_1^{⑦} = 0$$

类似地还有沿 α_2 边界上的等式以及角点上 $M_{12}^{⑦} = 0$。

4 用位移变分法求临界载荷

将(22)式和(23)式代入(25)式可得变形位能的应变分量表示式为

$$A'' = \iint \Big\{ \frac{E\delta}{2(1-u^2)} \{ e_{11}^{⑦2} + e_{22}^{⑦2} + 2\mu e_{11}^{⑦} e_{22}^{⑦} + \frac{1-\mu}{2}(e_{12}^{⑦} + e_{21}^{⑦})^2 + \frac{\delta^2}{12}[k_{11}^{⑦2} + k_{22}^{⑦2} + 2\mu k_{11}^{⑦} k_{22}^{⑦} + \frac{1-\mu}{2}(k_{12}^{⑦} + k_{21}^{⑦})^2]\} + \frac{1}{2}T_1^0(e_{11}^{⑦2} + e_{12}^{⑦2} + e_{13}^{⑦2}) + \frac{1}{2}T_2^0(e_{21}^{⑦2} + e_{22}^{⑦2} + e_{23}^{⑦2}) + T_{12}^0(e_{11}^{⑦} e_{21}^{⑦} + e_{12}^{⑦} e_{22}^{⑦} + e_{13}^{⑦} e_{23}^{⑦}) \Big\} A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{34}$$

将上式和(28)式代入(31)式所表示的弹性稳定问题的公式使有可能在解决具体问题时应用变分学的直接方法。对壳体来说，应变分量是指 e_{11}、e_{12}、e_{21}、e_{22}，而转动角是指 e_{13}、e_{23}。因此在和 1 相比时可略去 $e_{11}^{⑦}$、$e_{12}^{⑦}$、$e_{21}^{⑦}$、$e_{22}^{⑦}$和 $e_{13}^{⑦}$、$e_{23}^{⑦}$的平方项而仅保留 $e_{13}^{⑦}$ 和 $e_{23}^{⑦}$。将(28)式和(34)式代入(31)式简化后可得适用的变分公式为

$$\delta \iint \Big\{ \frac{E\delta}{2(1-\mu^2)} \{ e_{11}^{⑦2} + e_{22}^{⑦2} + 2\mu e_{11}^{⑦} e_{22}^{⑦} + \frac{1-\mu}{2}(e_{12}^{⑦} + e_{21}^{⑦})^2 + \frac{\delta}{12}[k_{11}^{⑦2} + k_{22}^{⑦2} + 2\mu k_{11}^{⑦} k_{22}^{⑦} + \frac{1-\mu}{2}(k_{12}^{⑦} + k_{21}^{⑦})^2]\} + \frac{1}{2}T_1^0 e_{13}^{⑦2} + \frac{1}{2}T_2^0 e_{23}^{⑦2} + T_{12}^0 e_{13}^{⑦} e_{23}^{⑦} \Big\} A_1 A_2 \, d\alpha_1 \, d\alpha_2 + \iint [q_n(e_{13}^{⑦} \delta u_1 + e_{23}^{⑦} \delta v_1) - (q_1 e_{13}^{⑦} + q_2 e_{23}^{⑦}) \delta w_1] A_1 A_2 \, d\alpha_1 \, d\alpha_2 = 0 \tag{35}$$

相应地，在稳定问题的平衡方程和边界条件中也可作同样的简化。

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