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模同态广义逆的一些结果*

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摘 要: 设 $\mathscr{U}$ 为结合环(含单位元 1), M 为左 $\mathscr{U}$- 模。本文考察模同态的广义逆,并用模同态的正则逆对模进行了分类,我们分别给出了直内射模,不可分解模及强不可分解模的充分必要条件。

关键词: 广义逆, Matlis 问题; 模

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Some Results of the Generalized Inverses of Module Homomorphisms

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Abstract: Let $\mathscr{U}$ be an associative ring with 1, and M a left $\mathscr{U}$- module. We consider the modules by the generalized inverses of homomorphisms, and we classify the modules with the regular inverses of homomorphisms. We make some necessary and sufficient conditions for a module to be direct injective, indecomposable, strongly indecomposable respectively.

Key words: generalized inverse; Matlis problem; module

1 INTRODUCTION

The principle of least squares was introduced by Legendre [1] and Gauss[2] to specifically handle the problem of inconsistent systems. A very important role played by the generalized inverse is providing best approximate solution to inconsistent linear systems. In fact, let $A\in C^{m\times n}$, $b\in C^m$, then a vector x is a leastsquares solution of $Ax = b$ if and only if

$$Ax = AA^{(1,3)}b$$

Moreover, the general least squares solution is

$$x = A^{(1,3)}b + (In - A^{(1,3)}A)y$$

with $A^{(1,3)}\in A\{1,3\}$ and arbitrary $y\in C^n$, where C is the complex number field. In fact, the idea of the generalized inverses has been applied not only the finite matrices but also to operators in various abstract settings. (See, for example, [3], [4].) In [5], Daniel L. Davis and Donald W. Robinson introduced the generalized inverses of morphisms. With the generalized inverse of morphism, they showed that the Axiom of choice is equivalent to the condition which every morphism $\Phi: X\rightarrow Y$ of $\mathscr{F}$ is regular, where $\mathscr{F}$ is a concrete category of nonempty sets and nonempty mappings. In this paper, we consider the modules by the generalized inverses of homomorphisms. We first handle the modules with the regular inverses of endomorphisms. Then, we classify the modules with the regular inverses of endomorphisms. Throught this paper, $\mathscr{U}$ will denote an associative ring with identity and M a unitary left $\mathscr{U}$- module.

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2 $\mathscr{U}$- MODULES AND THE REGULAR INVERSES OF $\mathscr{U}$- HOMOMORPHISMS

Let $\mathscr{U}$ be an associative ring with 1, M, N(left) $\mathscr{U}$- modules. A $\mathscr{U}$- homomorphism $f: M \rightarrow N$ is said to be regular if there exists a homomorphism $g: N \rightarrow M$ such that $fgf = f$. Unfortunately, most $\mathscr{U}$ - homorphisms which are of interest are not regular homomorphisms. However, we have the following result (see also[5])

Lemma 1. Let $\mathscr{U}$ be an associative ring with identity, $f: M \rightarrow N$ an $\mathscr{U}$- homomorphism. Then f is regular if and only if Kerf is a direct summand of M and Imf is a direct summand of N.

A $\mathscr{U}$- homomorphism g: $N \rightarrow M$ is called a regular inverse of $f: M \rightarrow N$ if it satisfies $fgf = f$. That is, g is a generalized 1- inverse of f. We denote $g \in f\{1\}$. A $\mathscr{U}$- module M is called a decomposable module if M can be decomposed the direct sum of two proper submodules. Otherwise M is called an indecomposable module. We are now able to state and prove the following result.

Theorem 1. Let M be a $\mathscr{U}$- module and $≊ \in End(M)$. Then the following statements are equivalent:

(i) M is an indecomposable module;

(ii) ≊ is regular if and only if either ≊ is 0 or ≊ is a unit;

(iii) The set of all non- zero regular endomorphisms of M is the group End^{*}(M).

Proof (i)⇒(ii), Let M be an indecomposable module. If $≊: M \rightarrow M$ is regular, then there exists $\Psi: M \rightarrow M$ such that $≊ \Psi ≊ = ≊$. It follows from Lemma 1 that ker ≊ and Im ≊ are direct summands of M.

If we let $≊ \neq 0$, then Ker $≊ \neq$ M and $Im ≊ \neq 0$. Since M is an indecomposable module, We know that $Ker ≊ = 0$ and $Im ≊ = M$. Therefore ≊ is an automorphism.

Conversely, if ≊ = 0 or ≊ is an automorphism, then ≊ is reguler, since $0 \cdot 0 \cdot 0 = 0$ and $≊ ≊^{-1} ≊ = 1 \cdot ≊ = ≊$.

(ii)⇒(iii) is obvious.

(iii)⇒(i) Assume M is a decomposable module, then there exists M_j such that $M = M_1 \oplus M_2$, $M_j \neq 0$ and $M_j \neq M$ $(j = 1, 2)$. Let $P_1: M \rightarrow M_1$ be the canonical projection, and $i_1: M_1 \rightarrow M$ the canonical injection. Then $i_1 p_1 \in End(M)$ and $(i_1 p_1) \cdot (i_1 p_1) \cdot (i_1 p_1) = (i_1 p_1)$. *Since* $i_1 p_1 \neq 0$ it follows form (iii) that $(i_1 p_1) \in End^{*}(M)$. That is, $i_1 p_1$ is a unit. But the endomorphism $i_1 p_1$ is not a unit, since $i_1 p_1$ is not a surjective $\mathscr{U}$- homomorphism. Therefore M is an indecomposable module.

We denote by $R_F(M)$ the regular set $R(M) \cup \{0\}$ of End (M), and denote $R(M)$ the set of all non - zero regular endomorphisms of M. For example, Z can be considered to be a Z- module, Z is an indecomposable module, then $R_F(Z) \cong Z_2$.

We say that a ring $\mathscr{U}$ is connected if the topological space Spec $\mathscr{U}$ is connected, this is the same as saying that the only idempotents in $\mathscr{U}$ are 0 and 1. By the previous theorem, we have the following consequence[6, P. 63].

Corollary 1. If M is a $\mathscr{U}$ module, then M is an indecomposable module if and only if End(M) is connected.

A strongly indecomposable module is an indecomposable module. Let M be an injective indecomposable module and J be the Jacobson radical of endomorphism ring End(M). Since End(M) is a local ring, J is the unique maximal ideal of End (M). Thus we have

Corollary 2. If M is a injective indecomposable module, then $R(M) = End(M) - J$.

Note that simple modules are indecomposable, but conversely it is not true. In fact, a simple module is a strongly indecomposable module. For example, Z is an indecomposable Z- module, but Z is not a strongly indecomposable Z- module. Moreover, Z is not a simple module.

Corollary 3. If M is a simple module, then $R_F(M)$ is a division ring and $R_F(M) = End(M)$.

Let M be a 𝒰- module. If ≊ : $M \to M$, ≊ ! $M \to M$ are endomorphisms such that ≊ ≊ ≒ = ≊, then it is not necessarily true that ≊ ≒ ≊ ≐ ≊ !. But if Ψ= ≊ ≒ ≊ !, then ≊ Ψ≊ = ≊ and Ψ≊ Ψ= Ψ. Moreoer, if ≊ ∈$R_F(M)$, ≊ ∈≊ {1} , then it is not necessarily true that Φ∈R_F(M) from Lemma l, However, there exists a Ψ∈Φ{1} such that Ψ∈$R_F(M)$. In general, if ≊ ∈$R_F(M)$, Ψ∈$R_F(M)$, then it is not necessarily true that

(a) ≊ + Ψ∈$R_F(M)$,

(b) ≊ Ψ∈$R_F(M)$,

In particular, if M is a simple module, it follows from corollary 3 that the two cases above are true, and $R_F(M)$ is a division ring.

Corollary 2. The following statements 'for a 𝒰- module M are equivalent:

(i) M is a strongly indecomposable module;

(ii) $R(M) = End^*(M)$, *and* $(R^c(M), +)$ is a semigroup, where $R^c(M) = End(M) - R(M)$.

Proof. (i)⇒(ii). By Theorem 1 (iii) and corollary 2.

(ii)⇒(i). Assume that $R(M) = End^*(M)$ and $R^c(M)$ is a semigroup. Let ≊ ∈End(M), Ψ∈$R^c(M)$, *if* ≊ Ψ∈$R(M)$, then ≊ Ψ is a unit. It follows that Ψ(≊ Ψ)$^{-1}$≊ ∈$R(M)$, since Ψ(≊ Ψ)$^{-1}$≊ · Ψ(≊ Ψ)$^{-1}$≊ Ψ(≊ Ψ)$^{-1}$≊ = Ψ(≊ Ψ)$^{-1}$≊. Also, it follows that Ψ(≊ Ψ)$^{-1}$≊ · Ψ(≊ Ψ)$^{-1}$≊ = Ψ(≊ Ψ)$^{-1}$≊, then it follows that Ψ(≊ Ψ)$^{-1}$≊ = 1, since Ψ(≊ Ψ)$^{-1}$≊ ∈$R(M)$. Obviously (≊ Ψ)$^{-1}$≊ Ψ= 1, thus Ψ is a unit, Ψ∈$R(M)$. That is impossible. Therefore $R^c(M)$ is an ideal of $End(M)$, that is, M is a strongly indecomposable module.

3 FURTHER RESULTS AND APPLICATIONS

Note that $End^*(M) \subseteq R(M) \subseteq End(M)$, $End^*(M) \cup \{0\} \subseteq R_F(M) \subseteq End(M)$, then we can classify the 𝒰- modules by $R(M)$ and $R_F(M)$. In the previous results, we considered the case $R_F(M) = End^*(M) \cup \{0\}$. Now we consider the case $R_F(M) = End(M)$.

Proposition 1. Let M be a 𝒰- module. If $R_F(M) = End(M)$, *then* $F_{dR_F(M)}M = 0$, *where* $F_{dR_F(M)}M$, is the flat dimension of $R_F(M)M$.

Proposition 2. If M is a semi- simple 𝒰- module, then $R_F(M) = End(M)$.

The proof of propositions above are easy, so we omit it.

Since every subspace of a finite- dimensioned vector space V is a direct summand of V, thus we have the following result.

Corollary 4. If V is a n - dimensional vector spaces over a division ring, then $R_F(V) = End(M)$.

As an example we consider the linear transformations of an - dimensional vector space V over the field of complex numbers, we consider the matrices $C^{n\times n}$ as the linear transformations $V \to V$, then we have the following corollary.

Corollary 5. Every $A \in C^{n\times n}$, there exists a $B \in C^{n\times n}$ such that $ABA = A$.

Proposition 3 Let M be a projective 𝒰- module. If $R_F(M) = End(M)$, then every a ∈M, there exists submodule Ia such that Ia is a direct summand of M, where 0≠I△𝒰.

Proof. By Lemmal and the Dual Basis theorem, the result can be obtained easily.

We denote $\underline{Emo(M)} = \{f: N \to M\}$ f isa monomorphism, N is direct summand of M}. Also, we denote $\overline{Emo(M)} = \{f: N \to M\}$ $\hat{f}$ N = f, $\hat{f}$ $_{N^\perp}$ = 0, f ∈$Emo(M)$}. Then it follows that $End^*(M) \subseteq \overline{Emo(M)} \subseteq End(M)$. In case $R_F(M) = \overline{Emo(M)}$, we have the following result.

Theorem 3. The following statements for a 𝒰- module M are equivalent:

(i) M is a direct injective module;

(ii) $R_F(M) = \overline{Emo(M)}$.

Proof. (i) ⇒ (ii) Let M be a direct injective module, introduced by W. K. Nicholson [see7]. If ≊ ∈ $R_F(M)$, then it follows from Lemmal that ker≊ is a direct summand of M. It follows that ≊ = $\hat{\Psi}$, $\hat{\Psi}_{ker≊} = 0$, $\hat{\Psi}_{ker≊^{\perp}}$ = ≊. Then it follows that ≊ ∈ $\overline{Emo(M)}$. That is $R_F(M) \subseteq \overline{Emo(M)}$. Conversely, assume $\hat{f} \in \overline{Emo(M)}$, then $\hat{f}_N = f$, $\hat{f}_{N^{\perp}} = 0$, where N is a direct summand of M, $f \in Emo(M)$. Since M is a direct injective module, it follows that there exists $g: M \to M$ such that $gf = i_N$. Moreover, it follows that $\hat{f}g\hat{f}_N = fi_N = f = \hat{f}_N$, $\hat{f}g\hat{f}_{N^{\perp}} = 0 = \hat{f}_{N^{\perp}}$. Then $\hat{f}g\hat{f} = \hat{f}$. That is, $\hat{f} \in R_F(M)$, $\overline{Emo(M)} \subseteq R_F(M)$. Therefore $R_F(M) = \overline{Emo(M)}$.

(ii) ⇒ (i). Let $g: N \to M$ be a monomorphism, where N is a direct summand of M. It follows from $R_F(M) = \overline{Emo(M)}$ that $\hat{g}$ is regular. Thus there exists $\Phi: M \to M$ such that $\hat{f}\Phi\hat{g} = \hat{g}$. Moreover, it follows that $gp_N\Phi gP_N = gp_N$, where P_N is the canonical projection of M onto N. Since g is a monomorphism (left cancellabel) and P_N is an epimorphism (right cancellable), it follows that $P_N\Phi g = i_N$. Then it follows that $i_N p_N \Phi g = i_N \cdot i_N = i_N$. Let $\bar{g}$ be $i_{N_p} N\Phi$, then $\bar{g}$ is a homomorphism from M to M such that $\bar{g}g = i_N$. It follows from [7] that M is a direct injective module. Since every injective module is a direct injective module we also have the following corollary.

Corollary 6. If M is an injective module, then $R_F(M) = \overline{Emo(M)}$.

Suppese $X = A_1 Ý A_2 Ý \cdots Ý A_n$ and $X = B_1 Ý B_2 Ý \cdots Ý B_n$. If $R(A_i)$, $R(B_j)$ satisfy that $R(A_i) = End^*(A_i)$, $R(B_j) = End^* B(j)$ and $R^c(B_j)$ is closed under the addition operation, then it follows from Theorem 1 and Theorem 2 that $n = m$, and after reindexing, $A_i \cong B_i$. Now we apply the results of the preceding section to the problem of Matlis of Krull− Schmidt′s theorem. [see 8] It is known that if the endomorphism ring of a $\mathscr{U}$- module M is a semiregular ring, then the problem of Matlis has an affirmative answer [see 9]. Since every VN regular ring is a semiregular ring, we have the following corollary.

Corollary 7. If M satisfies that $R_F(m) = End(M)$, then the problem of Matlis has an affirmative answer.

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