国防科技大学学报
JOURNAL OF NATIONAL UNIVERSITY OF DEFENSE TECHNOLOGY
第14卷第1期 1992年3月 Vol. 14 No. 1

任意三角域上 C^1 有理插值的一种新算法

方 逵 孙庆文

(系统工程与应用数学系)

摘 要 本文描述了任意三角域上 C^1 有理插值的一种组合算法，其特点是构造简单，计算方便，插值函数与文[3][4]比较，有着较高的逼近精度。最后给出了插值式的代数精度集和计算实例。

关键词 计算几何，三角域，插值函数，算子，奇点，多元逼近

分类号 O186

三角域上的插值与逼近在多元逼近论、有限元、曲面造型中有着广泛的应用。这方面有大量的文献[1-4]。本文研究在任意三角域上直接对已知函数及法向导函数的插值逼近。这种方法不需作仿射变换，所以构造简单，计算方便。本文构造了一种三次有理插值格式，这个插值式比文 [3] [4] 中的插值式有着更高的逼近精度，并且对插值式出现的奇点作了详细的讨论。

1 C^1 有理插值函数 $G(x,y)$ 的构造

设 BC 边的三角形 ABC 的最长边，以 BC 边为 y 轴构成直角坐标系如图 1.

令 $x_0=|OA|$，$y_1=|OB|$，$y_2=-|OC|$，

$$m=(1-x/x_0)y_j \qquad j=\begin{cases}1 & y\geq 0\\ 2 & y<0\end{cases}$$
$$n=(1-y/y_j)x_0$$

显然 $y=m$，$x=n$ 分别是线段 AB，AC 的方程。

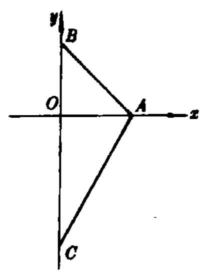

图 1

为方便起见，用 T 表示三角形 ABC，∂T 表示 T 的边界，并设 $F(x,y)$是 T 上的被插函数。

下面引入两个插值算子 P_1,P_2.

$$\begin{cases}P_1F(x,y)=\sum_{i=0}^{1}\phi_i(x/n)n^iF_{i0}(0,y)+\sum_{i=0}^{1}\varphi_i(x/n)n^iF_{i0}(n,y)\\ P_2F(x,y)=\sum_{i=0}^{1}\phi_i(y/m)m^iF_{0i}(x,0)+\sum_{i=0}^{1}\varphi_i(y/m)m^iF_{0i}(x,m)\end{cases} \qquad (1)$$

其中，$\phi_0(t)=(t-1)^2(2t+1)$，$\phi_1(t)=t(t-1)^2$，$\varphi_0(t)=t^2(3-2t)$，$\varphi_1(t)=(t-1)t^2$，F_{ij}

* 1991年1月7日收稿 1991年6月20日收修改稿

$=\partial F^{(i+j)}(x,y)/\partial x^i\partial y^j$

易验证 $P_1F(x,y)$，$P_2F(x,y)$具有如下性质：

$$\begin{cases} P_1F(x,y)|_{\partial T} = F(x,y)|_{\partial T}, & \left.\dfrac{\partial P_1F(x,y)}{\partial x}\right|_{\partial T} = \left.\dfrac{\partial F(x,y)}{\partial x}\right|_{\partial T} \\ \left.\dfrac{\partial P_1F(x,y)}{\partial y}\right|_{\partial T} = \left.\dfrac{\partial F(x,y)}{\partial y}\right|_{\partial T} \end{cases} \tag{2}$$

$$\begin{cases} P_2F(x,y)|_{AB\cup AC\cup OA} = F(x,y)|_{AB\cup AC\cup OA} \\ P_2F(x,y)/\partial x|_{AB\cup AC\cup OA} = \partial F(x,y)/\partial x|_{AB\cup AC\cup OA} \\ P_2F(x,y)/\partial y|_{AB\cup AC\cup OA} = \partial F(x,y)/\partial y|_{AB\cup AC\cup OA} \end{cases} \tag{3}$$

这里注意到 $\partial P_1F/\partial y$ 在 OA 上不连续。下面利用这两个“部分”插值函数构造在 ∂T 上插值 $F(x,y)$及 F 的一阶偏导数的函数 $G(x,y)$。文[3][4]中仅用到(1)式中的一个插值算子，因此逼近精度要低些。这一节将证明 $G(x,y)$的奇点是可去奇点，从而论证了 $G(x,y)\in C^1(T)$.

设 $$\alpha(x,y) = y^3/(x^2+y^3), \quad \beta(x,y) = x^2/(x^2+y^3)$$

令 $$G(x,y) = \alpha(x,y)P_1F(x,y) + \beta(x,y)P_2F(x,y) \tag{4}$$

由插值性质(2),(3)，易得

$$G(x,y)|_{AB\cup AC} = (\alpha(x,y)+\beta(x,y))F(x,y)|_{AB\cup AC}$$
$$= F(x,y)|_{AB\cup AC}$$
$$G(x,y)|_{BC} = G(x,y)|_{x=0},\ G(x,y)|_{BC} = F(x,y)|_{BC}$$

且 $$G(x,y)|_{OA} = G(x,y)|_{y=0} = F(x,y)|_{OA}$$

综上可得 $$G(x,y)|_{\partial T\cup OA} = F(x,y)|_{\partial T\cup OA} \tag{5}$$

同理可证 $$\left.\frac{\partial G(x,y)}{\partial x}\right|_{\partial T} = \left.\frac{\partial F(x,y)}{\partial x}\right|_{\partial T}$$
$$\left.\frac{\partial G(x,y)}{\partial x}\right|_{OA} = \left.\frac{\partial F}{\partial y}\right|_{OA}, \left.\frac{\partial G(x,y)}{\partial x}\right|_{OA} = \left.\frac{\partial F(x,y)}{\partial x}\right|_{OA} \tag{6}$$

由式(5)、(6)知，$G(x,y)$在 $\partial T|_{OA}$上插值 $F(x,y)$及 $F(x,y)$的一阶偏导数。但是从(1)和(4)式注意到：$(x_0,0)$是 $P_1F(x,y)$，$\partial P_1F(x,y)/\partial x$，$\partial P_1F(x,y)/\partial y$ 的奇点；$(0,y_1)$，$(0,y_2)$是 $P_2F(x,y)$，$\partial P_2F(x,y)/\partial x$，$\partial P_2F(x,y)/\partial y$ 的奇点；$(0,0)$是 $G(x,y)$，$\partial G(x,y)/\partial x$，$\partial G(x,y)/\partial y$ 的奇点；$(x_0,0)$，$(0,y_1)$，$(0,y_2)$可由文[5]中方法证明是可去奇点。下面只证明$(0,0)$是可去奇点。

设 $$L(x,y) = P_1F(x,y) - P_2F(x,y)$$
$$M(x,y) = \partial L(x,y)/\partial x, N(x,y) = \partial L(x,y)/\partial y$$
$$P(x,y) = \partial^2 L(x,y)/\partial x^2,\ Q(x,y) = \partial^2 L(x,y)/\partial x\partial y$$
$$R(x,y) = \partial^2 L(x,y)/\partial y^2$$

则

$$\begin{cases} G(x,y) = P_2F(x,y) + \dfrac{y^3}{x^2+y^3}L(x,y) \\ \partial G(x,y)/\partial x = \partial P_2F(x,y)/\partial x + \dfrac{y^3}{x^2+y^3}M(x,y) - \dfrac{2xy^3}{(x^2+y^3)^2}L(x,y) \\ \partial G(x,y)/\partial y = \partial P_2F(x,y)/\partial y + \dfrac{x^2}{x^2+y^3}N(x,y) - \dfrac{3x^2y^2}{(x^2+y^3)^2}L(x,y) \end{cases} \quad (7)$$

将(0,0)代入(2)，(3)式得：$L(0,0)=M(0,0)=N(0,0)$

又 $L(x,y)$，$M(x,y)$，$N(x,y)$在(0,0)附近泰勒展开：

$$L(x,y) = L(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)L(x,y)|_{(0,0)} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 L(x,y)|_{(\theta_1 x,\theta_1 y)}$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 L(\theta_1 x,\theta_1 y)$$

$$M(x,y) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)M(\theta_2 x,\theta_2 y)$$

$$N(x,y) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)N(\theta_3 x,\theta_3 y)$$

其中 $0<\theta_1,\theta_2,\theta_3<1$.

设 $F(x,y)\in C^3(T)$，则 $L(x,y)$，$M(x,y)$，$N(x,y)$，$P(x,y)$，$Q(x,y)$，$R(x,y)$都是 T 上的有界函数。于是有下列的近似表示：

$$\begin{cases} L(x,y) = O(x^2) + O(xy) + (O(y^2) \\ M(x,y) = O(x) + O(y),\ N(x,y) = O(x) + O(y) \end{cases} \quad (8)$$

将式(8)代入式(7)得

$$\begin{cases} G(x,y) = P_2F(x,y) + O(x^2) + O(xy) + O(y^2) \\ \partial G(x,y)/\partial x = \partial P_2F(x,y)/\partial x + O(x) + O(y) + O(\sqrt{y}) \\ \partial G(x,y)/\partial y = \partial P_2F(x,y)/\partial y + O(x) + O(y^2) + O(y\sqrt{y}) \end{cases} \quad (9)$$

对式(9)取极限并注意到式(3)有

$$\lim_{(x,y)\to(0,0)} G(x,y)=F(0,0),$$

$$\lim_{(x,y)\to(0,0)} \partial G(x,y)/\partial x = \lim_{(x,y)\to(0,0)} \partial P_2F(x,y)/\partial x = F_{10}(0,0)$$

$$\lim_{(x,y)\to(0,0)} \partial G(x,y)/\partial y = \lim_{(x,y)\to(0,0)} \partial P_2F(x,y)/\partial y = F_{01}(0,0)$$

于是定义

$$G(0,0) = F(0,0), G_{10}(0,0) = F_{10}(0,0), G_{01}(0,0) = F_{01}(0,0)$$

到此证明了(0,0)是可去奇点。

由于 $G(x,y)$，$\partial G(x,y)/\partial x$，$\partial G(x,y)/\partial y$ 在 T 上除去可去奇点外都是连续的，故 G

$(x,y)\in C^1(T)$.

定理 1 设 $F(x,y)\in C^3(T)$，则

$$G(x,y)=\frac{y^3}{x^2+y^3}P_1F(x,y)+\frac{x^2}{x^2+y^3}P_2F(x,y)$$

在 $\partial T\cup OA$ 上插值 $F(x,y)$及其偏导数，且 $G(x,y)\in C^1(T)$.

2 误差估计

设 $F\in C^6(T)$，由于 $G(x,y)$是 $F(x,y)$的插值函数，且

$$F(x,y)-G(x,y)=\alpha(x,y)(F(x,y)-P_1F(x,y))+\beta(x,y)(F(x,y)-P_2F(x,y)) \tag{10}$$

由式(1)对每一个固定的 $y\in[y_2,y_1]$，则 $P_1F(x,y)$是 x 的三次多项式，且在点$(0,y_1)$，(n,y)处插值 $F(x,y)$，$\partial F(x,y)/\partial y$. 所以由 Hermite 插值余式有

$$|F(x,y)-P_1F(x,y)|\le x^2(x-n)^2M_1 \tag{11}$$

其中 $M=\max\limits_{(x,y)\in T}\{F_{40}(x,y)\}$.

同理可得

$$|F(x,y)-P_2F(x,y)|\le y^2(y-(1-x/x_0)y_1)^2(y-(1-x/x_0)y_2)^2M_2 \tag{12}$$

其中 $M_2=\max\limits_{(x,y)\in T}\{F_{06}(x,y)\}$.

将式(11)，(12)代入式(10)得

$$\begin{aligned}|F(x,y)-G(x,y)|&\le\frac{y^3}{x^2+y^3}|F(x,y)-P_1F(x,y)|+\frac{x^2}{x^2+y^3}|F(x,y)-P_2F(x,y)|\\&\le\frac{y^3}{x^2+y^3}x^2(x-n)^2M_1\\&\quad+\frac{x^2}{x^2+y^3}y^2(1-(1-x/x_0)y_1)^2(1-(1-x/x_0)y_2)^2M_2\end{aligned}$$

令 $$h=\max\{|BC|,|OA|\}$$

由极值原理易证

$$(x-n)^2y^3\le\frac{36}{3125}h^5$$

$$y^2(y-(1-x/x_0)y_1)^2(y-(1-x/x_0)y_2)^2\le\frac{1}{16}h^6$$

综上有

$$\begin{aligned}|F(x,y)-G(x,y)|&\le\frac{36}{3125}h^5M_1+\frac{M_2}{16}h^6\\&=O(h^5)\end{aligned} \tag{13}$$

定理 2(误差估计定理) 设 $F\in C^6(T)$，则 $G(x,y)$插值 $F(x,y)$的精度是 $O(h^5)$.

由式(13)中 M_1，M_2 的表达式易求得 $G(x,y)$的代数精度集为

$$\tau=\{x^iy^j:i+j\le3\}.$$

3 实 例

设 $F(x,y)=\sin(2\pi(1-y))$，下面是用本文方法计算的最大偏差结果表。

表 1

三角形顶点	BBG 的最大偏差	(4)的最大偏差
$A(0,0),B(1,0),C(0,1)$	0.5776	0.0045632
$A(0,0),B\left(\frac{1}{2},0\right),C\left(0,\frac{1}{2}\right)$	0.00328	0.0000243
$A(0,0),B\left(\frac{1}{4},0\right),C\left(0,\frac{1}{4}\right)$	0.00019	0.0000012
$A(0,0),B\left(\frac{1}{8},0\right),C\left(0,\frac{1}{8}\right)$	0.000004	0.0000008

这里 BBG 的最大偏差来源于文[4]，上表说明本文的结果明显优于 [4] 中的结果。

参 考 文 献

1 Schumaker L L. In: Chui c k and schumaker L L edited. Approximation Theory. Academic press, 1976: 203～268

2 Barnhill R E. In: Mathematical Software Ⅱ, Edited by Rice J R. Academic Press. 1979: 69～161

3 方逵等. 工程数学学报. 1987, 4 (2): 59～66

4 Wang C Y. CAD. 1983, 15 (1): 33～36

An New Algorithm for C^1 Rational Interpolation Over an Arbitrary Triangle

Fang Kui　　Suen Qinwen

(Department of System Engineering and Applied Mathematics)

Abstract

Combinational algorithm for C^1 rational interpolation function over an arbitrary triangle is described in this paper. The interpolation is a convex combination of two functions. The construction of the interpolation is simple and the calculations involved is convenient, The precision of the interpolation is better than those in [3－4]. Finally, the precision set is obtained, and a calculation example is given.

Key words　operator, singularity, computational geometry, triangle; interpolation function